How Strong is the Weak Axiom?*

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Abstract

We characterize those abstract choice problems for which the satisfaction of the weak axiom of revealed preference suffices for the strong rationalizability of any choice correspondence. Roughly, this requires that all circuits on a certain graph defined from the budget collection of the choice problem are able to be broken in an intuitive way. The condition is non-monotone, and is satisfied by both very small and very large budget collections. We additionally provide a notion of local integrability for an abstract choice correspondence, and prove an ordinal variant of the Hurwicz-Uzawa integrability theorem that holds in the full generality of the abstract choice model. We fully characterize how complete the domain of a choice correspondence must be for the weak axiom and local integrability to jointly guarantee strong rationalizability.

1 Introduction

A common theme in choice and demand theory is that, for sufficiently rich data sets, the possible inconsistencies of the data set with preference maximization may be characterized by simple conditions that are independent of the larger-scale structure of the data. When

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one observes a complete enough collection of choices by an individual, cyclic choice patterns over many alternatives necessarily also imply cycles amongst choices over only a few. This paper is concerned with what constitutes a 'complete enough' set of observations for such results to obtain, independent of the specific choices observed.

An example is the manner in which the structure of a choice problem affects the power of the weak axiom. Consider four alternatives $\{a, b, c, d\}$, and suppose an individual is presented with choices between $\{a, b\}$, $\{b, c\}$, $\{c, d\}$, and $\{d, a\}$. If this individual were to choose a in the presence of b, b in the presence of c and so forth cyclically, her choice behavior would be consistent with the weak axiom. This is because her choice behavior contains no preference *reversal*. However, it would be inconsistent with preference maximization, as it would violate the generalized axiom: it contains a *cycle*. On the other hand, if the agent were additionally presented with choices over $\{a, b, c\}$, and $\{c, d, a\}$, the presence of the cycle from her other choices would necessarily force her to make a preference reversal: given a was chosen in the presence of b, and b was chosen in the presence of c, the only choice from $\{a, b, c\}$ that does not itself constitute a reversal is to choose a, revealing it to be preferred to c. But, as c was chosen in the presence of d and d in the presence of a, *every* possible choice from $\{c, d, a\}$ constitutes a reversal. Though this collection of budgets is far from complete, the budgets nevertheless intersect in such a manner as to force any revealed preference cycle to necessarily induce a concomitant revealed preference reversal. Were these choice sets selected by an experimenter to be presented to the individual, the experimental setup would preclude the existence of testable implications of preference transitivity beyond pairwise coherent choice.

The primary contribution of this paper is to formally characterize the observational richness requirements for abstract choice problems under which the entire content of strong rationalizability reduces to the satisfaction of the weak axiom. Section 3 provides necessary and sufficient conditions on the budget collection of an abstract choice problem for the weak axiom to suffice for the generalized axiom of revealed preference. This amounts to a requirement that, if the choice problem contains a collection of budgets on which an agent could choose cyclically over some set of alternatives, that there is some budget contained in the union of this collection that suitably covers those alternatives. Section 4 considers a similar problem through the lens of integrability theory. We introduce a weak local 'integrability' condition akin to discrete, ordinal Slutsky or Antonelli symmetry for abstract choice correspondences. We then provide necessary and sufficient conditions on how rich the domain of a choice correspondence must be for (i) the weak axiom, and (ii) local integrability to exhaust the testable implications of rational choice. The richness condition takes the form of a combinatorial-topological 'no holes' requirement for a geometric object representing the domain of the choice correspondence. Finally, combining these results, we show the budget condition is necessary and sufficient for (i) the local integrability of every choice correspondence that obeys the weak axiom, and (ii) the domain of observations to have 'no holes' in the above sense. We interpret this as a decomposition of the budget richness condition into its local and global implications.

2 Related Literature

A well-known result due to Arrow (1959) (see also Sen (1971)) demonstrates that in the abstract choice model, if one observes choice from *all* two- and three-element subsets of the space of alternatives, the entirety of the hypothesis of rational choice is subsumed by the satisfaction of the weak axiom.[1] The demand integrability theory of Samuelson, Hurwicz, Uzawa and others (Samuelson (1950), Hurwicz and Uzawa (1971)) says that, if one has choice data from *every* linear budget, then the hypothesis that the observed demand is induced by constrained-optimal choice according to a well-behaved utility function is wholly characterized by the weak axiom plus a local integrability condition. In all these instances, however, it is less well understood precisely how rich the set of observations need be for the global consistency conditions guaranteeing rationalizability to collapse to simply-verifiable pairwise or local ones.

[1]Similarly in spirit, a cardinal variant of the problem of when the weak and generalized axioms coincide is an area of research in mechanism design, where it is of great interest for which type spaces the (weak) monotonicity and cyclic monotonicity of an allocation rule coincide. See, for example, Saks and Yu (2005), Ashlagi et al. (2010), or Archer and Kleinberg (2014).

The question of when the weak axiom is empirically distinguishable from the strong in the classical demand framework has a long history. Rose (1958) first proved that the weak and strong axioms coincide in the case of two goods, though Gale (1960) soon after established that this result did not hold for the case of three or more goods. In a recent contribution, Cherchye et al. (2018) characterized those linear budget collections for which several variants of the weak and strong axioms coincide for the classical demand model. Interestingly, they find that many widely used price-consumption datasets have large subsets exhibiting insufficient price variation to independently distinguish the weak and strong axioms.[2] Given the apparent empirical shortcomings of field data for purposes of independently testing these phenomena, one is naturally led to consider how to construct simple, finite, laboratory experiments capable of rectifying this deficiency. It is empirically and computationally desirable then to understand the problem for those environments in which one must take seriously indivisibilities, price non-linearities, or other economic phenomena contrary to the linear budget paradigm.

There has also been longstanding interest in the measurement of the degree of the deviation from rationality of a given choice data set (e.g. Houtman and Maks (1985), Varian (1990), Echenique et al. (2011), Apesteguia and Ballester (2015)). For such purposes, it is not only of paramount importance to be able to efficiently compute all violations of the generalized axiom present in a given data set, but also to understand the dependencies between violations. For many choice problems, subsets of alternatives have the property that if choices between elements of the subset are cyclic, then those choices necessarily also induce other revealed preference cycles elsewhere in the data. Unaccounted for, this can lead to potential misestimation of the magnitude of the deviation from rationality of a data set. Our results clarify how the structure of the underlying choice problem creates dependencies between possible violations of the generalized axiom.

[2] They find that roughly 70% of the Spanish survey ECPF (Encuesta Continua de Presupestos Familiares) panel dataset (see, for example Beatty and Crawford (2011)) satisfies their condition for when a WARP-based analysis is equally informative as SARP-based. Even more drastically, roughly 97% of price triples in the British FES (Family Expenditure Survey) cross-sectional data set (see, for example, Blundell et al. (2003), Blundell et al. (2008), Blundell et al. (2015)) satisfy their condition for WARP and SARP to coincide.

Finally, it is common to characterize theories of behavioral rationality under a 'full domain' hypothesis, whereby choice is specified over every non-empty subset of alternatives (e.g. Manzini and Mariotti (2007), Bernheim and Rangel (2009), Masatlioglu et al. (2012)). Assumptions of this form are useful to theorists, as they empower the weak axiom, or suitable variants thereof, to characterize the acyclicity of the binary relations of interest. For such models, De Clippel and Rozen (2014) seek to understand empirically just what can be tested under more realistic, limited data. Conversely, this paper studies just how far the full domain hypothesis may be relaxed in the specification of the underlying model, while still permitting a sufficiently powerful weak axiom.

3 The Structure of the Budget Collection

3.1 Notation and Definitions

Let X be an arbitrary set of **alternatives** from which an agent chooses. Let $\Sigma \subseteq 2^X \setminus \{\varnothing\}$ be a collection of **budgets** encoding the specifics of the collection of constraints under which choice occurs. When $\Sigma = 2^X \setminus \{\varnothing\}$, we say that Σ is **complete**. We refer to the tuple (X, Σ) as an abstract **choice problem**. For any subset $A \subseteq X$, we define the restriction of Σ to A as those elements of Σ wholly contained in A:

$$\Sigma|_A = \big\{B \in \Sigma : B \subseteq A\big\},$$

and for a collection of subsets $\mathcal{A} \subseteq 2^X$, it will be useful to define the shorthand:

$$\Sigma|_{\mathcal{A}} = \left\{B \in \Sigma : B \subseteq \bigcup_{A \in \mathcal{A}} A\right\}.$$

A mapping $c : \Sigma \to 2^X \setminus \{\varnothing\}$ is a **choice correspondence** if and only if it satisfies:

$$(\forall B \in \Sigma) \ \ c(B) \subseteq B.$$

Let $\mathcal{C}(X, \Sigma)$ denote the collection of all choice correspondences for the problem (X, Σ). Given some $c \in \mathcal{C}(X, \Sigma)$, a weak order $\succeq$ on X **strongly rationalizes** c if:

$$(\forall B \in \Sigma) \ \ c(B) = \big\{x \in B : \forall y \in B, x \succeq y\big\}.$$

Given a $c \in \mathcal{C}(X, \Sigma)$, its revealed preference pair $(\succsim_c, \succ_c)$ is defined via: $x \succsim_c y$ if there exists some $B \in \Sigma$ such that $x, y \in B$ and $x \in c(B)$, and $x \succ_c y$ if there exists some $B \in \Sigma$ such that $x, y \in B$, $x \in c(B)$ and $y \notin c(B)$.

A choice correspondence $c \in \mathcal{C}(X, \Sigma)$ satisfies the **weak axiom** of revealed preference if $x \succsim_c y$ implies $\neg\, y \succ_c x$.[3] We say c obeys the **generalized axiom** of revealed preference if $(\succsim_c, \succ_c)$ is acyclic.[4] Denote the collections of $c \in \mathcal{C}(X, \Sigma)$ that obey the weak axiom by $\mathcal{W}(X, \Sigma)$, and that obey the generalized axiom by $\mathcal{G}(X, \Sigma)$. In particular, it was shown by Richter (1966), making use of an extension theorem due to Szpilrajn (1930), that a choice correspondence is strongly rationalizable by a weak order if and only if it obeys the generalized axiom.[5] In light of this, we will interchangeably refer to the satisfaction of the generalized axiom as strong rationalizability.

3.2 Results

For purposes of combinatorial bookkeeping, it will be helpful to define an auxiliary structure that, for a given choice problem (X, Σ), encodes precisely which pairs of alternatives it is even possible for a preference to be revealed between. Let $\Gamma(X, \Sigma)$ be an undirected graph whose vertex set is X, and whose edge-set E_Γ is given by the relation of two vertices belonging to some common budget:

$$\{x, y\} = e_{xy} \in E_\Gamma \iff \exists B \in \Sigma \text{ s.t. } \{x, y\} \subseteq B.$$

[3] The characterization of precisely which choice correspondences satisfy the weak axiom has been extensively studied by Wilson (1970). For a characterization of what kinds of binary relations the maximization of which is consistent with the weak axiom under general Σ, see Mariotti (2008).

[4] That is, if there are no *finite* cycles of the form $x_0 \succsim_c x_1 \succsim_c \cdots \succsim_c x_N \succ_c x_0$. It is without loss of generality to suppose that these alternatives are all distinct, as any cycle containing multiple appearances of the same alternative necessarily also contains a sub-cycle consisting only of distinct alternatives.

[5] We note, however, that Szpilrajn (1930) acknowledges the priority of Banach, Kuratowski, and Tarski in discovering, though not publishing, the result.

We term $\Gamma(X,\Sigma)$ the **budget graph**.[6] If $c \in \mathcal{W}(X,\Sigma)$, then for any $e \in E_\Gamma$ there is a well-defined (possibly empty) binary relation $\succsim_c |_e$.[7] Given a collection of edges $E' \subseteq E_\Gamma$, define:

$$\succsim_c |_{E'} = \bigcup_{e \in E'} \succsim_c |_e.$$

A loop in Γ is a finite subgraph $\gamma = (V_\gamma, E_\gamma)$ such that every vertex in V_γ belongs to precisely two edges in E_γ.

For a loop $\gamma \subseteq \Gamma(X,\Sigma)$, a collection of budgets $\mathcal{B}_\gamma \subseteq \Sigma$ is a **cyclic collection** for γ if, for every $e \in E_\gamma$ there exists a $B \in \mathcal{B}_\gamma$ with $e \subseteq B$. Note that for every loop in Γ, by definition there exists at least one cyclic collection. Given a loop γ and cyclic collection $\mathcal{B}_\gamma$, we say $\mathcal{B}_\gamma$ is **covered** if either:

(i) There exists a $\bar{B} \in \Sigma|_{\mathcal{B}_\gamma}$ such that $V_\gamma \subseteq \bar{B}$; or

(ii) There exists a $\bar{B} \in \Sigma|_{\mathcal{B}_\gamma}$ such that $\bar{B}$ contains a pair of elements of V_γ that are not connected by any edge in E_γ.

Note that condition (i) implies (ii) if and only if $|V_\gamma| > 3$.

Lemma 1. *Let (X,Σ) be a choice problem and let γ be a loop in $\Gamma(X,\Sigma)$. Then there exists choice correspondence $c \in \mathcal{W}(X,\Sigma)$ such that $\succsim_c |_{E_\gamma}$ is a cycle if and only if there exists a cyclic collection $\mathcal{B}_\gamma$ and choice correspondence $\tilde{c} \in \mathcal{W}(X,\Sigma|_{\mathcal{B}_\gamma})$ such that $\succsim_{\tilde{c}} |_{E_\gamma}$ is a cycle.*

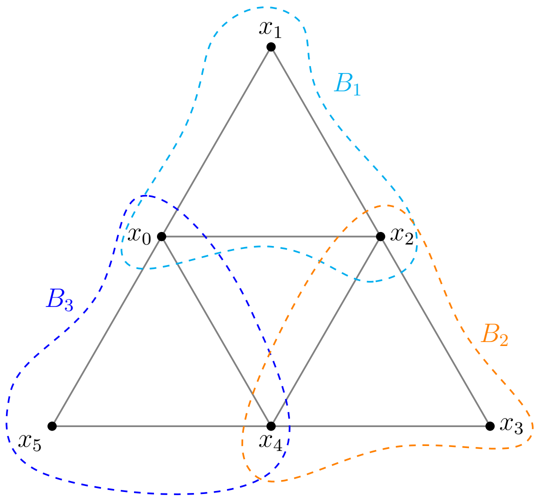
Lemma 1 is a reduction of domain result, and shows that all of the obstructions to the existence of a choice correspondence whose revealed preference contains a cycle supported on γ involve only choices made from budgets contained in the restriction of Σ to the various cyclic collections for γ. The next lemma fully characterizes the existence of 3-cycles, the

[6] Equivalently, the edge relation of $\Gamma(X,\Sigma)$ is equal to the irreflexive kernel of the revealed preference induced by the complete indifference choice function or, alternatively, the irreflexive kernel of the union of every revealed preference induced by a choice function obeying the weak axiom.

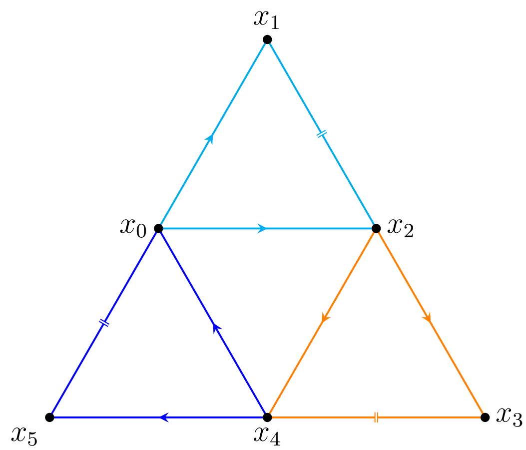
[7] This is because an edge in any graph is itself a two-element subset of the graph's vertex set, thus:

$$\succsim_c |_{e_{xy}} = \succsim_c \cap \{x,y\} \times \{x,y\}$$

is well-defined.



(a) A choice problem with six alternatives and three budgets. The sole cyclic collection for the outermost loop is covered.



(b) The revealed preference relation induced by a choice correspondence satisfying the weak axiom. In particular, the outermost loop supports a revealed preference cycle.

Figure 1: For loops of length greater than three, even if every cyclic collection for the loop is covered, there may still exist choice correspondences inducing a revealed preference cycle on the loop. All one may guarantee is that such a choice correspondence also induces a strictly shorter revealed preference cycle elsewhere, here the cycle on $\{x_0, x_2, x_4\}$.

minimal cycles of any revealed preference pair arising from a choice correspondence satisfying the weak axiom.

Lemma 2. *Let (X, Σ) be a choice problem and let γ be a loop in $\Gamma(X, \Sigma)$ with $|V_\gamma| = 3$. Then there exists a choice correspondence $c \in \mathcal{W}(X, \Sigma)$ with $\succsim_c |_{E_\gamma}$ a cycle if and only if there exists a cyclic collection $\mathcal{B}_\gamma$ that is not covered.*

Unfortunately, such a clean result does not obtain for longer loops. Loops of length greater than three may have the property that all of their cyclic collections are covered, and yet they still support preference cycles from choice correspondences that obey the weak axiom, as Example 1 shows.

Example 1. Let $X = \{x_0, \ldots, x_5\}$, and $\Sigma = \big\{\{x_0, x_1, x_2\}, \{x_2, x_3, x_4\}, \{x_4, x_5, x_0\}\big\}$. Then the budget graph is composed of three 3-cliques, one for each element of Σ. In particular, the loop $\gamma = (X, \{\{x_0, x_1\}, \ldots, \{x_4, x_5\}, \{x_5, x_0\}\})$ has a single cyclic collection given by all of Σ. This cyclic collection is covered, as any element of it contains a pair of points non-adjacent in γ, however there exists a choice correspondence $c \in \mathcal{W}(X, \Sigma)$ such that $\succsim_c |_{E_\gamma}$ is a preference cycle: $c(\{x_0, x_1, x_2\}) = \{x_1, x_2\}$, $c(\{x_2, x_3, x_4\}) = \{x_3, x_4\}$, and $c(\{x_4, x_5, x_0\}) = \{x_5, x_0\}$. ▌

However, it turns out that if every cyclic collection of a loop is covered, then the existence of a revealed preference cycle supported on the loop implies the existence of a strictly shorter loop elsewhere that also supports a revealed preference cycle. In Example 1, this takes the form of the 3-cycle induced by c over $\{x_0, x_2, x_4\}$. See also Figure 1.

Lemma 3. *Let (X, Σ) be a choice problem and let γ be a loop in $\Gamma(X, \Sigma)$ with $|V_\gamma| > 3$. Suppose there exists a choice function $c \in \mathcal{W}(X, \Sigma)$ with $\succsim_c |_{E_\gamma}$ a cycle. If every cyclic collection $\mathcal{B}_\gamma$ is covered, then there exists a loop γ' in $\Gamma(X, \Sigma)$ such that $|V_{\gamma'}| < |V_\gamma|$ and with $\succsim_c |_{E_{\gamma'}}$ a cycle.*

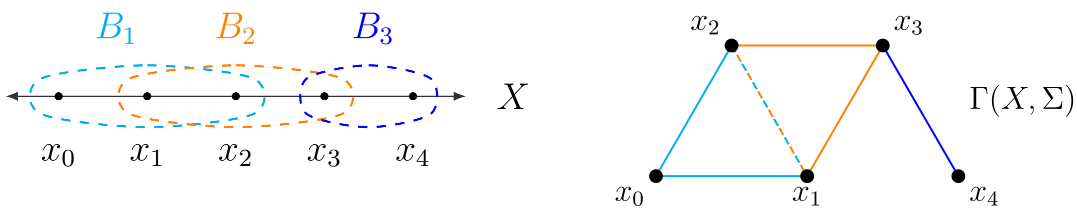
Call a budget collection Σ **well-covered** if, for every loop γ in the budget graph $\Gamma(X, \Sigma)$, every cyclic collection $\mathcal{B}_\gamma$ for γ is covered. Well-coveredness is a condition purely on the underlying problem (X, Σ), without reference to any choice data. The primary result of this section is that the well-coveredness of Σ is both necessary and sufficient for the weak axiom of revealed preference to coincide with strong rationalizability for *any* choice correspondence.

Theorem 1. *Let (X, Σ) be a choice problem. Then $\mathcal{W}(X, \Sigma) = \mathcal{G}(X, \Sigma)$ if and only if Σ is well-covered.*

If Σ is well-covered, it need not be the case that $\Sigma' \subset \Sigma$ is also well-covered.[8] The property of being well-covered does obey a particular form of monotonicity, however.

Proposition 1. *For all $A' \subseteq A \subseteq X$, if $\Sigma|_A$ is well-covered, then $\Sigma|_{A'}$ is well-covered.*

[8]This may be seen by considering a three-element set of alternatives $X = \{x_0, x_1, x_2\}$. If the budget collection Σ consists precisely of the three two-element subsets of X, then Σ is not well-covered. However, if we remove any set from Σ, it vacuously becomes well-covered: its budget graph becomes a tree. Alternatively, if we add X as a budget to Σ, then the resulting budget collection becomes well-covered.



(a) A choice problem with a linearly ordered set of alternatives, and budgets consisting of order intervals.

(b) The budget graph associated to the choice problem. Every loop in the graph necessarily contains some order-minimal vertex; this vertex has the property that, for any cyclic collection, it and both of its neighbors are contained in the same budget, implying the cyclic collection is covered.

Figure 2: When X is a linearly ordered set and all budgets consist of order-intervals, then Σ is always well-covered. An identical argument holds for weakly ordered sets of alternatives.

Proof. Suppose, for purposes of contraposition, that there exists some uncovered cyclic collection $\mathcal{B}_\gamma \subseteq \Sigma|_{A'}$ for some loop γ in $\Gamma(X, \Sigma|_{A'})$. This cyclic collection cannot become covered by passing to $\Sigma|_A$, as any covering budget $B \in \Sigma|_A$ must, by definition, be contained in the union of the elements of $\mathcal{B}_\gamma$, each of which are contained in A', forcing B itself to necessarily belong to $\Sigma|_{A'}$. $\square$

This allows for applications of Theorem 1 to budget collections that are not well-covered. Suppose one wishes to detect whether a given choice correspondence c exhibits preference cycles within some subset $A \subseteq X$. If $\Sigma|_A$ is well-covered, and the agent's choices obey the weak axiom, it suffices then to only search for cycles amongst those choices over cyclic collections that do not belong to $\Sigma|_A$.

Additional structure on the choice problem problem may often be used to verify (or refute) the well-coveredness of a given budget collection. The following examples use an intrinsic order structure on X to rephrase well-coveredness of Σ as a matter of the budget collection being suitably 'aligned.' This is illustrated in Figure 2.

Corollary 1. *Let $(X, \leq)$ a weakly ordered set, and suppose Σ consists only of order-intervals of X. Then $\mathcal{W}(X, \Sigma) = \mathcal{G}(X, \Sigma)$.*

Proof. Consider an arbitrary loop γ in the budget graph with vertex set $V_\gamma = \{x_0, \ldots, x_n\}$, ordered such that the edge set is of the form $\{x_i, x_{i+1}\}$, indices taken modulo $(n+1)$. Suppose that x_i is any $\leq$-minimal element in V_γ. Then in particular, for the triple of adjacent points x_{i-1}, x_i, x_{i+1} it follows that:

$$x_i \leq x_{i-1}, x_{i+1}.$$

As $\leq$ is complete, without loss of generality suppose $x_{i-1} \leq x_{i+1}$. Then for any cyclic collection for γ, there exists some budget containing the pair $\{x_i, x_{i+1}\}$; since budgets are order-intervals, any such budget contains x_{i-1}, and hence the cyclic collection is covered. As the cyclic collection was arbitrary, we conclude Σ is well-covered. $\square$

The strong intermediate value property of order-intervals underpinning the preceding corollary can be relaxed only at the expense of assumptions elsewhere.[9]

Corollary 2. *Let $(X, \leq_X)$ be a lattice, and suppose further that:*

(i) Σ contains only totally ordered subsets of X,

(ii) every pair of elements Σ is comparable in the strong set order.

Then $\mathcal{W}(X, \Sigma) = \mathcal{G}(X, \Sigma)$.

Proof. Let γ denote an arbitrary loop in $\Gamma(X, \Sigma)$. By (i) we conclude that every edge pair of γ is totally ordered. As $\leq_X$ is a partial order, $\leq_X |_{E_\gamma}$ admits a local minimum, in the sense of the existence of $x_{i-1}, x_i, x_{i+1} \in V_\gamma$ such that:

$$x_i <_X x_{i-1}, x_{i+1},$$

and $\{x_{i-1}, x_i\}, \{x_i, x_{i+1}\} \in E_\gamma$. Let $\mathcal{B}_\gamma$ be an arbitrary cyclic collection for γ. In light of (ii), without loss of generality let $B_{x_{i-1}x_i} \leq_{SSO} B_{x_i x_{i+1}}$ for two budgets in $\mathcal{B}_\gamma$ with $B_{x_{i-1}x_i}$ containing $\{x_{i-1}, x_i\}$ and $B_{x_i x_{i+1}}$ containing $\{x_i, x_{i+1}\}$. Then by the strong set order:

$$x_{i-1} = x_i \vee x_{i-1} \in B_{x_{i+1}x_i},$$

and hence $B_{x_{i+1}x_i}$ covers $\mathcal{B}_\gamma$. As γ and $\mathcal{B}_\gamma$ were arbitrary, we again conclude that Σ is well-covered. $\square$

[9] Corollary 1 is false, for example, when Σ is only required to consist of totally ordered sets. For example, $X = \{x_0, x_1, x_2\}$, where elements are linearly ordered by their subscripts and $\Sigma = \{\{x_0, x_1\}, \{x_1, x_2\}, \{x_0, x_2\}\}$ is not well-covered.

4 Abstract Choice and Integrability

4.1 Classical Integrability Theory

The problem of the integrability of a demand, or, dually, budgeter vector field has been well studied (e.g. Ville (1946), Samuelson (1950), Ville and Newman (1951), Hurwicz (1971), Hurwicz and Uzawa (1971), Hurwicz and Richter (1979b), Afriat (2014)). It is a standard result that under suitable differentiability hypotheses, the integrability of a budgeter function is equivalent to the negative semi-definiteness of the Antonelli matrix (corresponding to the weak axiom, see Kihlstrom et al. (1976)) and the absence of Ville cycles: closed curves in the commodity space, oriented such that the path is always moving in a direction of increasing revealed preference (see Ville and Newman (1951), Hurwicz and Richter (1979a), Hurwicz and Richter (1979b)). An analogue of this result was proven in Richter (1966) for the revealed preference: a revealed preference pair may be extended by a preference relation if and only if it obeys the generalized axiom, hence obeys the weak axiom, and exhibits no longer cycles.[10] [11] The absence of Ville cycles, however, is a global property of the budgeter vector field, and instead, the conventional characterization of integrability is phrased in terms of the local data of the problem: the negative semi-definiteness and symmetry of the Antonelli matrix. The equivalence between the everywhere symmetry of the Antonelli matrix of a budgeter vector field and the local absence of Ville cycles was proven in Hurwicz and Richter (1979b).[12] By an 'integrability' theorem, we then refer to any result passing from a local absence of revealed preference cycles into global acyclicity over the whole space of alternatives. Classically, such extensions rely upon the presumption of a complete domain, whereby the budgeter function is presumed to be observed over the entire space of alternatives. One then exploits the fact that this set is convex and hence topologically trivial in a suitable sense to obtain a global

[10]See also Hansson (1968).

[11]We take the perspective that a revealed preference cycle of three or more distinct elements is nothing more than a discrete, ordinal Ville cycle.

[12]Formally, Theorem 1.c in Hurwicz and Richter (1979b) states:

> *'The Antonelli matrix symmetry axiom of a C^1 budgeter function holds on some neighborhood of a point $y \in X$ if and only if some neighborhood of y contains no C^∞ Ville cycles.'*

This motivates our discussion of Antonelli (or, dually, Slutsky) symmetry as a local no cycles condition.

result.

This section will examine large-scale implications of the structure of Σ for the rationalizability of choice correspondences. We provide an appropriate, purely combinatorial notion of Antonelli symmetry for the abstract choice correspondences that guarantees their local integrability. We show that when no completeness axiom is imposed upon Σ, the familiar conditions of (i) the weak axiom and (ii) local integrability need not jointly suffice for strong rationalizability. This is a consequence of Σ potentially being too 'sparse' for such results to obtain. We fully characterize when such traditional integrability results hold, even absent the assumption of complete Σ, and provide a geometric interpretation of how the structure of Σ permits or prohibits these local revealed preference conditions from extending.

4.2 An Abstract Integrability Theorem

Let (X, Σ) be a fixed choice problem, with $\Gamma(X, \Sigma)$ its budget graph. Let:

$$T_\Gamma = \big\{\{x, y, z\} \subseteq X : \{x, y\}, \{y, z\}, \{x, z\} \in E_\Gamma\big\}.$$

The combinatorial **domain** associated to the choice problem (X, Σ) is the triple $\mathcal{D}(X, \Sigma) = (X, E_\Gamma, T_\Gamma)$. The combinatorial domain of a choice problem essentially serves as a 'triangulation' of the set X using only the information encoded in the budget graph. For any $c \in \mathcal{W}(X, \Sigma)$, we refer to c as **locally rationalizable** if there exists an order-extension $\succeq_c$ of its revealed preference $\succsim_c$ whose restriction to the vertex set of each 3-clique in the budget graph is complete and transitive:

$$(\forall T \in T_\Gamma) \quad \succeq_c|_T \text{ is complete and transitive.}$$

Local rationalizability is necessary, though not sufficient, for the strong rationalizability of c (as any strong rationalization is also a local rationalization). Particularly, we interpret the local rationalizability of c as a discrete, ordinal analogue of Antonelli symmetry for abstract choice correspondences. It says only that we may pass to a (still possibly very incomplete) extension of $\succsim_c$ that obeys a no local Ville cycles condition, with local being interpreted as about any triangle in the budget graph.[13] The fact that one must potentially consider an extension of $\succsim_c$ is simply a consequence of allowing for the possibility that Σ is highly incomplete and c does not reveal any preference between some pairs in some T_Γ.[14]

The sufficiency of the weak axiom and local rationalizability for the acyclicity of a revealed preference $\succsim_c$ is determined wholly by the structure of $\mathcal{D}(X, \Sigma)$. Intuitively, the denser the budget graph, the larger T_Γ, and hence the stronger the implication of local rationalizability. We now turn to the clarification of the precise combinatorial structure needed. Let $\tilde{T}$ be a subset of T_Γ. Let $\tilde{X}$ denote the points of X contained within some element of $\tilde{T}$, and $\tilde{E}$ the subset of edges in E_Γ contained in some element of $\tilde{T}$. Then the **sub-domain** generated by $\tilde{T}$ is defined as the tuple:

$$\mathcal{D}(X, \Sigma)|_{\tilde{T}} = (\tilde{X}, \tilde{E}, \tilde{T}).$$

For finite $\tilde{T}$, we say that $\mathcal{D}(X, \Sigma)|_{\tilde{T}}$ is a **simple** sub-domain if it is:

(i) **Combinatorially Trivial**: If T and T' are elements of $\tilde{T}$, there is a *unique* sequence of distinct elements of $\tilde{T}$:

$$T = T_1, T_2, \ldots, T_k = T'$$

such that, for all $1 \leq j \leq k - 1$, T_j and T_{j+1} have precisely a pair of elements in common.

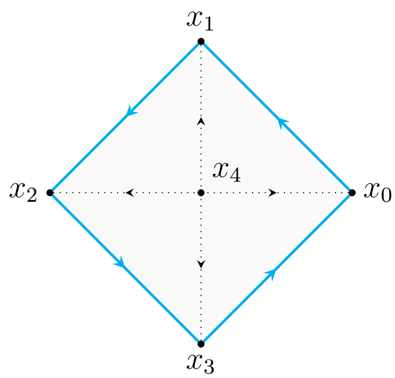
(ii) **Topologically Trivial**: The sub-domain $\mathcal{D}(X, \Sigma)|_{\tilde{T}}$ has a first Betti number of zero.[15]

Combinatorial triviality of a sub-domain requires the elements of $\tilde{T}$ to 'fit together' in a particularly elementary manner: if one imagines an undirected graph whose nodes are

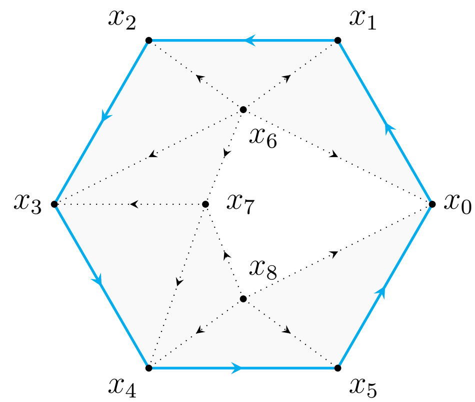
[13] In the classical case, Antonelli symmetry does not imply the weak axiom, whereas here, local rationalizability as defined above requires the satisfaction of the weak axiom by the revealed preference pair restricted to each triangle $T \in T_\Gamma$. We do not consider this a meaningful loss of generality, as for our purposes of clarifying the economic interpretation of well-coveredness we are concerned exclusively with choice correspondences that already obey the weak axiom.

[14] If, for example, $E_\Gamma \subseteq \Sigma$, there would be no need to consider an extension.

[15] Any (sub-)domain is a simplicial complex so its simplicial homology is well-defined. Topological triviality then simply requires the vanishing of the one-dimensional simplicial homology with real coefficients of $\mathcal{D}(X, \Sigma)|_{\tilde{T}}$, i.e. $H_1(\mathcal{D}(X, \Sigma)|_{\tilde{T}}; \mathbb{R}) = 0$. See Munkres (1984) p. 34.



(a) A cyclic revealed preference (blue) with a locally rational extension on a topologically trivial domain that is not combinatorially trivial. The failure of combinatorial triviality permits the possibility of cyclic but nonetheless locally rational extensions.

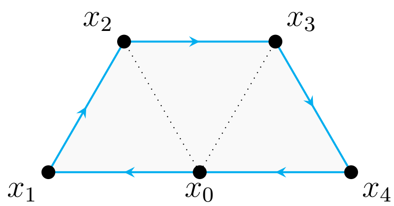


(b) A cyclic revealed preference with a locally rational extension on a combinatorially trivial domain. The domain however retracts onto its outer boundary loop, hence it is topologically non-trivial. The failure of topological triviality again allows for the possibility of cyclic but locally rational extensions.

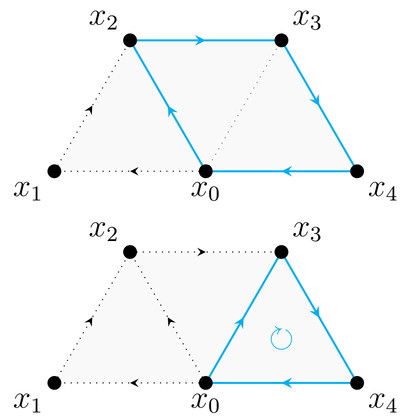
Figure 3: On non-simple domains, the weak axiom and local rationalizability need not imply strong rationalizability.

the elements of $\tilde{T}$ and whose edge relation is given by sharing a two-element subset, then combinatorial triviality amounts to asking this graph be a tree. Topological triviality is a simply-connected type condition that requires $\mathcal{D}(X,\Sigma)|_{\tilde{T}}$ to have no holes in it. Any reader unfamiliar with the formalism underlying the definition of simplicial homology is invited to simply interpret topological triviality precisely as "no holes" in an intuitive sense, with no great risk of running astray. Figure 3 illustrates the independence of the two triviality conditions, and shows neither may be omitted if one wishes locally rational binary relations to be acyclic.

By slight abuse of notation, we call a domain $\mathcal{D}(X,\Sigma)$ simple if, for every loop $\gamma \subseteq \Gamma(X,\Sigma)$, γ is contained in some simple sub-domain of $\mathcal{D}(X,\Sigma)$. On a simple domain, if a choice correspondence (i) obeys the weak axiom, and (ii) is locally rationalizable, then it is



(a) A cyclic binary relation, indicated in blue, on a set of five alternatives. We seek to extend it to a locally rational relation on the simple domain consisting of the three indicated triangles.



(b) For either the left (or right) triangle, the extension is uniquely determined. However, adding this extra relation induces a new, shorter cycle, eventually precluding the existence of a locally rational extension.

Figure 4: On simple domains, local rationality implies acyclicity. For a cyclic revealed preference, on any simple sub-domain containing the loop supporting the cycle, the locally rational extension is uniquely determined on 'leaf' triangles, and this unique extension induces a strictly shorter cycle on the same simple sub-domain.

also strongly rationalizable.[16] An example of this is given in Figure 4. The simplicity of the domain, it turns out, is the weakest possible completeness assumption on Σ under which *any* such traditional integrability result can possibly hold: on non-simple domains there always exist choice correspondences which obey the weak axiom and are locally rationalizable, yet nonetheless are not strongly rationalizable.

[16]Note that for any set X, $\mathcal{D}(X, 2^X \setminus \{\varnothing\})$ is simple, hence this subsumes the case where Σ is complete.

Theorem 2. *Let (X, Σ) be a choice problem with $\mathcal{D}(X, \Sigma)$ a simple domain. Then a choice correspondence $c \in \mathcal{C}(X, \Sigma)$ is strongly rationalizable if and only if:*

(i) It obeys the weak axiom; and

(ii) It is locally rationalizable.

Moreover, (i) and (ii) are jointly equivalent to the strong rationalizability of c if and only if $\mathcal{D}(X, \Sigma)$ is simple.

It is worth emphasizing the parallels between the above results and the classical theory. Even without the smooth structure of Euclidean space, one still requires both the weak axiom and a suitable local no-cycles condition. However, when there is incomplete data, these necessary conditions may no longer suffice. What is needed in addition is the ability to observe a sufficiently rich sample of the agent's choices, characterized here by the simplicity of $\mathcal{D}(X, \Sigma)$.

While the reliance on algebraic topological properties in the definition of simplicity is expositionally unfortunate, we hope the strength and scope of Theorem 2 provides suitable ends to justify these means. A noteworthy consequence of these methods, is that one obtains a purely combinatorial integrability theorem, independent of *any* analytic, point-set, or order-theoretic assumptions on model primitives. This is particularly notable given the historical program of attempting to weaken the differentiability hypotheses of the classical integrability theory.[17] There is a cost to this generality, however. In the classical theory, while one supposes a great deal more structure on the budgeter or demand function and its domain, one obtains a rationalizing utility with commensurately fine properties. Comparatively, all that is guaranteed by Theorem 2 is a rationalizing weak order. This is the price, it appears, of an integrability theory that not only allows for finite data sets, but also imposes no hypotheses that are non-falsifiable by such data sets.[18]

[17] See, for example, Berger and Meyers (1966), Hartman (1970), and Berger and Myers (1971).

[18] Put another way, Theorem 2 operates wholly within the empirical content of revealed preference theory, in the sense of Chambers et al. (2014).

4.3 Relation to Well-coveredness

The integrability theorem of the preceding section clarifies the economic interpretation of the well-coveredness of Σ. On a local level, the well-coveredness of Σ implies that every choice correspondence that obeys the weak axiom necessarily is also locally rationalizable. This is akin to guaranteeing that every budgeter vector field with a negative semi-definite Antonelli matrix has a symmetric Antonelli matrix. However, well-coveredness also implies precisely the global structure of the domain $\mathcal{D}(X, \Sigma)$ needed for these local implications to suffice for strong rationalizability: simplicity. Indeed, it is characteristic of both these properties.

Theorem 3. *Let (X, Σ) be a choice problem. Then Σ is well-covered if and only if both:*

(i) $\mathcal{D}(X, \Sigma)$ is simple; and

(ii) If $c \in \mathcal{W}(X, \Sigma)$ then c is locally rationalizable.

This decomposition of the well-coveredness into its local and global implications begs the natural question of the budget characterization of local rationalizability. Lemma 2 may be interpreted as a characterization of when, for every $c \in \mathcal{W}(X, \Sigma)$ and for every $T \in T_\Gamma$,

$$\succsim_c |_T \text{ is acyclic.}$$

While clearly necessary, this is, however, insufficient to guarantee local rationalizability, as illustrated by the below example.

Example 2. Let $X = \{x_0, x_1, x_2, x_3, x_4\}$ and $\Sigma = \big\{\{x_1, x_2\}, \{x_2, x_3\}, \{x_3, x_4\}, \{x_4, x_1\}, \{x_0, x_1, x_2, x_3, x_4\}\big\}$. In light of the last budget, $\Gamma(X, \Sigma)$ is the complete graph on five vertices. Moreover, every cyclic collection of every loop of length three is covered. However, the loop $\{x_1, x_2\}, \{x_2, x_3\}, \{x_3, x_4\}, \{x_4, x_1\}$ has an uncovered cyclic collection composed of the first four budgets. There is a $c \in \mathcal{W}(X, \Sigma)$ that induces a cycle $x_1 \prec_c x_2 \prec_c x_3 \prec_c x_4 \prec_c x_1$ on this loop (and chooses $\{x_0\}$ from the final budget). Hence while $\{x_1, x_2\}, \{x_2, x_4\}, \{x_4, x_1\}$, for example, does not support any cycle, c is nonetheless not locally rationalizable, as any extension would be forced to have $x_4 \prec_c x_2$ from $T = \{x_1, x_2, x_4\}$, but similarly would be forced to have $x_2 \prec_c x_4$ from $T' = \{x_2, x_3, x_4\}$, an impossibility. ∎

The cardinality-constrained problem, in essence considered by Arrow (1959), provides an example of a special case in which the absence of 3-cycles does coincide with the local rationalizability of every choice correspondence. In what follows, suppose Σ contains no budget of cardinality greater than three, and denote the sub-collection of three-element budgets by $\Sigma_3 \subseteq \Sigma$. Then the weak axiom implies local rationalizability for the cardinality-constrained case if and only if the vertex set of every triangle in the budget graph is itself a budget in Σ_3.

Proposition. *Let (X, Σ) be a cardinality-constrained choice problem. Then every $c \in \mathcal{W}(X, \Sigma)$ is locally rationalizable if and only if:*

$$\mathcal{T}_\Gamma = \Sigma_3.$$

Notably, this holds independently of the structure of the domain $\mathcal{D}(X, \Sigma)$ of the choice problem. Making use of this, we additionally obtain the following necessary and sufficient version for the classical result of Arrow that if Σ contains *all* budgets of cardinality weakly less than three, then the weak axiom suffices for the acyclicity of the revealed preference.

Theorem 4. *Let (X, Σ) be a cardinality-constrained choice problem. Then $\mathcal{W}(X, \Sigma) = \mathcal{G}(X, \Sigma)$ if and only if $\mathcal{D}(X, \Sigma)|_{\Sigma_3}$ is simple.*

Conclusions

The objective of this paper is to weaken the informational requirements of choice theory. While this paper considers only the problem of strong rationalizability, we hope the ideas and tools developed within will find use more broadly. A great deal of revealed preference theory, both classical and behavioral, suppose an exhaustive collection of observations in the possession of the empiricist. This paper constitutes a first step in studying just how widely applicable such 'full domain' theories are in practice, and how they may be adapted to hold in as wide-ranging practical circumstances as possible.

Appendix I: Budget Proofs

Proof of Lemmas 1-3:

Lemma. *Let (X, Σ) be a choice problem and let γ be a loop in $\Gamma(X, \Sigma)$. Then there exists choice function $c \in \mathcal{W}(X, \Sigma)$ such that $\succsim_c |_{E_\gamma}$ is a cycle if and only if there exists a cyclic collection $\mathcal{B}_\gamma$ and choice function $\tilde{c} \in \mathcal{W}(X, \Sigma|_{\mathcal{B}_\gamma})$ such that $\succsim_{\tilde{c}} |_{E_\gamma}$ is a cycle.*

Proof. ($\Longrightarrow$): Suppose there exists a $c \in \mathcal{W}(X, \Sigma)$ such that $\succsim_c |_{E_\gamma}$ is a cycle. Then there exists some cyclic collection $\mathcal{B}_\gamma$ with the property that the choices inducing $\succsim_c |_{E_\gamma}$ are all made on elements of $\mathcal{B}_\gamma$. Then the restriction of c to $\Sigma|_{\mathcal{B}_\gamma}$ must still obey the weak axiom, and clearly satisfies the conclusion of the lemma.

($\Longleftarrow$): Suppose now there exists a cyclic collection $\mathcal{B}_\gamma$ and a $\tilde{c} \in \mathcal{W}(X, \Sigma|_{\mathcal{B}_\gamma})$ such that $\succsim_{\tilde{c}} |_{E_\gamma}$ is a cycle. Define an extension of $\tilde{c}$ to all of Σ as follows:

$$c(B) = \begin{cases} \tilde{c}(B) & \text{if } B \in \Sigma|_{\mathcal{B}_\gamma} \\ B \setminus \big(\cup_{\tilde{B} \in \mathcal{B}_\gamma} \tilde{B} \big) & \text{else.} \end{cases}$$

This defines a choice correspondence in $\mathcal{W}(X, \Sigma)$, for if $x \succsim_c y$ for distinct x, y, either $x, y \in \cup_{\tilde{B} \in \mathcal{B}_\gamma} \tilde{B}$, in which case there can be no violation of the weak axiom as $\tilde{c}$ is in $\mathcal{W}\big(X, \Sigma|_{\mathcal{B}_\gamma}\big)$, or $x \notin \cup_{\tilde{B} \in \mathcal{B}_\gamma} \tilde{B}$, in which case by construction $\neg\, y \succ_c x$, and thus $c \in \mathcal{W}(X, \Sigma)$. $\square$

Lemma. *Let (X, Σ) be a choice problem and let γ be a loop in $\Gamma(X, \Sigma)$ with $|V_\gamma| = 3$. Then there exists a choice correspondence $c \in \mathcal{W}(X, \Sigma)$ with $\succsim_c |_{E_\gamma}$ a cycle if and only if there exists a cyclic collection $\mathcal{B}_\gamma$ that is not covered.*

Proof. ($\Longleftarrow$): Suppose that $\mathcal{B}_\gamma$ is an uncovered cyclic collection for γ of minimal cardinality. Let us denote $E_\gamma = \{e_0, e_1, e_2\}$. Then, in particular, for every $e_j \in E_\gamma$, there is a unique $B_j \in \mathcal{B}_\gamma$ with $e_j \subseteq B_j$. Define $\tilde{c} \in \mathcal{C}(X, \Sigma|_{\mathcal{B}_\gamma})$ via:

$$\tilde{c}(B) = \begin{cases} e_j \cap e_{j+1} & \text{if } \exists\, e_j \in E_\gamma \text{ s.t. } B \cap V_\gamma = e_j \\ B \cap V_\gamma & \text{if } |B \cap V_\gamma| = 1 \\ B & \text{else.} \end{cases}$$

where all subscripts are taken mod-3. Note $\tilde{c}$ is well-defined, as $\mathcal{B}_\gamma$ is uncovered from which it follows the first two cases exhaust the possibilities for budgets in $\Sigma|_{\mathcal{B}_\gamma}$ that intersect V_γ. Moreover, $\tilde{c} \in \mathcal{W}(X, \Sigma|_{\mathcal{B}_\gamma})$. First, observe the restriction of the pair $(\succsim_{\tilde{c}}, \succ_{\tilde{c}})|_{E_\gamma}$ satisfies the weak axiom. But the only alternatives $\tilde{c}$ reveals strictly preferred to any others all lie in V_γ, and the only goods ever revealed preferred to elements of V_γ also lie in V_γ. Hence $\tilde{c} \in \mathcal{W}\big(X, B \in \Sigma|_{\mathcal{B}_\gamma}\big)$, and by Lemma 1 there exists a $c \in \mathcal{W}(X, \Sigma)$ such that $\succsim_c|_{E_\gamma}$ is cyclic.

($\Longrightarrow$): Let $c \in \mathcal{W}(X, \Sigma)$ be such that $\succsim_c|_{E_\gamma}$ is cyclic. Then there exists a cyclic collection $\mathcal{B}_\gamma$ on which choices generating the cycle $\succsim_c|_{E_\gamma}$ are made; fix such a collection. We now show that this cyclic collection must be uncovered, lest there exist some $B \in \Sigma|_{\mathcal{B}_\gamma}$ such that $V_\gamma \subseteq B$. Suppose, for sake of contradiction, that such a B exists.

Case 1: Suppose first that $c(B) \cap V_\gamma \neq \varnothing$. Then either $c(B)$ induces complete indifference across V_γ, or there exists some pair of elements of V_γ that is either strictly preferred to, or strictly dominated by the third element. Both possibilities preclude the existence of the cycle $\succsim_c|_{E_\gamma}$ for any $c \in \mathcal{W}(X, \Sigma)$.

Case 2: Suppose then that $c(B) \cap V_\gamma = \varnothing$: then for all $x \in V_\gamma$ and $y \in c(B)$ we have $y \succ_c x$. But $c(B) \subset B \subseteq \cup_{\tilde{B} \in \mathcal{B}_\gamma} \tilde{B}$, and since for all $x \in V_\gamma$ there exists some $\tilde{B}$ such that $x \in c(\tilde{B})$, there exists an $\tilde{x} \in V_\gamma$ and $\tilde{B} \in \mathcal{B}_\gamma$ such that $\tilde{x}, y \in \tilde{B}$ and $\tilde{x} \in c(\tilde{B})$. This contradicts our hypothesis that $c \in \mathcal{W}(X, \Sigma)$. □

Lemma. *Let (X, Σ) be a choice problem and let γ be a loop in $\Gamma(X, \Sigma)$ with $|V_\gamma| > 3$. Suppose there exists a choice correspondence $c \in \mathcal{W}(X, \Sigma)$ with $\succsim_c|_{E_\gamma}$ a cycle. If every cyclic collection $\mathcal{B}_\gamma$ is covered, then there exists a loop γ' in $\Gamma(X, \Sigma)$ such that $|V'_\gamma| < |V_\gamma|$ and with $\succsim_c|_{E_{\gamma'}}$ a cycle.*

Proof. Let $\mathcal{B}_\gamma$ be a minimal cyclic collection on which choices inducing $\succsim_c|_{E_\gamma}$ are made, and suppose $\mathcal{B}_\gamma$ is covered. Then there exists some $B \in \Sigma|_{\mathcal{B}_\gamma}$ such that B contains a non-adjacent pair of vertices of γ. We proceed in two cases.

Case 1: Suppose first that $c(B)$ does not intersect V_γ. Let $x_k, x_{k'} \in B \cap V_\gamma$ be one such non-adjacent pair of vertices, and let $y \in c(B)$. As $c(B) \subseteq B \subseteq \cup_{\tilde{B} \in \mathcal{B}_\gamma} \tilde{B}$, and $\mathcal{B}_\gamma$ is a minimal cyclic collection on which choices inducing the cycle $\succsim_c |_{E_\gamma}$ are made, there is some $\tilde{B}_{k^*} \in \mathcal{B}_\gamma$ containing y, such that there is some $x_{k^*} \in c(\tilde{B}_{k^*}) \cap V_\gamma$. Without loss of generality, let:

$$x_{k'} \succsim_c \cdots \succsim_c x_{k^*} \succsim_c \cdots \succsim_c x_k.$$

In particular, by our hypothesis that c obeys the weak axiom, we cannot have $x_{k^*} = x_k$ (or $x_{k'}$).[19] As $c(B)$ does not contain any element of V_γ by hypothesis, but $x_{k'} \in B$:

$$y \succ_c x_{k'},$$

and, as $x_{k^*}, y \in \tilde{B}_{k^*}$,

$$x_{k^*} \succsim_c y.$$

Thus:

$$y \succ_c x_{k'} \succsim_c \cdots x_{k^*} \succsim_c y.$$

Define γ' to be the graph with $V_{\gamma'}$ given by the above collection of points, and $E_{\gamma'}$ consisting of those pairs related in the above cycle (clearly as there is a non-empty revealed preference for each pair this forms a loop in $\Gamma(X, \Sigma)$). By construction, $\succsim_c |_{E_{\gamma'}}$ is a cycle. Now, since $x_{k^*} \neq x_k$, $x_k \notin V_{\gamma'}$. Moreover, since x_k and $x_{k'}$ are non-adjacent in γ, under $\succsim_c |_{E_\gamma}$ we also have:

$$x_k \succsim_c \cdots \succsim_c \bar{x} \succsim_c \cdots \succsim_c x_{k'}$$

along the 'other side' of the loop. Thus we also have that $\bar{x} \notin V_{\gamma'}$. So while we have added a point y not in V_γ to our $V_{\gamma'}$, we have omitted at least two others, x_k and $\bar{x}$, and we conclude:

$$|V_{\gamma'}| < |V_\gamma|$$

as required.

Case 2: Suppose now that $c(B)$ intersects V_γ. As B contains the non-adjacent pair $x_k, x_{k'} \in V_\gamma$, the only way that $c(B)$ can avoid revealing a preference between x_k and $x_{k'}$ is if neither is in but both are adjacent in γ to $c(B)$. Moreover, this argument holds for every non-adjacent

[19] As $y \succ_c x_k$ and $y \succ_c x_{k'}$ by hypothesis, but $x_{k^*} \succsim_c y$ via choice on B_{k^*}.

pair of vertices of γ contained in B. Now, if $c(B)$ induces a revealed preference $x_i \succsim_c x_j$ between any pair of non-adjacent vertices $x_i, x_j \in V_\gamma$ this partitions $\succsim_c |_{E_\gamma}$ into two sub-cycles, one of which must always contain a strict relation (either from $\succsim_c |_{E_\gamma}$ or resulting from a strict revealed preference between x_i and x_j). Letting γ' be defined by the vertices and pairs supporting any such sub-cycle suffices to prove the claim. Thus suppose that $c(B)$ does not induce any revealed preference between any non-adjacent pair (lest we be done). Thus $c(B)$ is adjacent to both x_k and $x_{k'}$ (and hence singleton) and $c(B) = \{x^*\}$ induces:

$$x_k \prec_c x^* \succ_c x_{k'}.$$

But these three points are all elements of V_γ, hence by virtue of $\succsim_c |_{E_\gamma}$ being a cycle we have either:

$$x_k \succsim_c x^* \succsim_c x_{x'}$$

or the reverse. But both of these yield contradiction via a violation of the weak axiom, and hence there exists a strictly shorter $\succsim_c$-cycle. □

Proof of Theorem 1:

Theorem. *Let (X, Σ) be a choice problem. Then $\mathcal{W}(X, \Sigma) = \mathcal{G}(X, \Sigma)$ if and only if Σ is well-covered.*

Proof. ($\Longleftarrow$): For purposes of contraposition, suppose that $\mathcal{W}(X, \Sigma) \neq \mathcal{G}(X, \Sigma)$. Then there exists some loop γ in the budget graph $\Gamma(X, \Sigma)$ and some choice correspondence $c \in \mathcal{W}(X, \Sigma)$ such that $\succsim_c |_{E_\gamma}$ is a cycle. If $|V_\gamma| = 3$, then by Lemma 2, Σ is not well-covered and we are done. Hence suppose γ is of length strictly greater than three. Then there exists some cyclic collection $\mathcal{B}_\gamma$ on which choices generating the cycle $\succsim_c |_{E_\gamma}$ are made. If $\mathcal{B}_\gamma$ is not covered, we are done, hence suppose it is. Then by Lemma 3 there exists a loop γ' in the budget graph of strictly shorter length such that $\succsim_c |_{E_{\gamma'}}$ is also a cycle. As we have already concluded this process cannot repeat until it hits a three-cycle, we conclude that at some stage, there exists some loop $\gamma^{(n)}$ for which there exists a cyclic collection $\mathcal{B}_{\gamma^{(n)}}$ which is not covered and hence Σ is not well-covered.

($\Longrightarrow$): We again proceed by contraposition. If a cyclic collection for a budget graph loop of length 3 is uncovered, by Lemma 2, we immediately obtain $\mathcal{W}(X, \Sigma) \neq \mathcal{G}(X, \Sigma)$. Suppose then there exists some loop γ with $|V_\gamma| > 3$ with a cyclic collection $\mathcal{B}_\gamma$ that is uncovered (without loss of generality, let $\mathcal{B}_\gamma$ be a minimal such uncovered cyclic collection) In particular, let $E_\gamma = \{e_0, \dots, e_{J-1}\}$. By virtue of γ being uncovered, for each $e_j \in E_\gamma$ there exists a $\tilde{B}_j \in \mathcal{B}_\gamma$ such that for all $j \in \{0, \dots, J-1\}$:

$$e_j = \tilde{B}_j \cap V_\gamma,$$

and by the minimality of $\mathcal{B}_\gamma$, these $\{\tilde{B}_j\}$ are unique and completely exhaust $\mathcal{B}_\gamma$. Furthermore, for all $B \in \Sigma|_{\mathcal{B}_\gamma}$, $B \cap V_\gamma$ necessarily also either equals some e_j, is singleton, or is empty.[20] Thus, letting (subscripts taken mod-J):

$$\tilde{c}(B) = \begin{cases} e_j \cap e_{j+1} & \text{if } \exists\, e_j \text{ s.t. } e_j = B \cap V_\gamma \\ B \cap V_\gamma & \text{if } |B \cap V_\gamma| = 1 \\ B & \text{else,} \end{cases}$$

we obtain a choice correspondence $c \in \mathcal{W}(X, \Sigma|_{\mathcal{B}_\gamma})$ by an argument identical to that in the proof of Lemma 2, only for a longer cycle. Clearly $\succsim_{\tilde{c}} |_{E_\gamma}$ is cyclic and by Lemma 1 this extends to a choice correspondence in $c \in \mathcal{W}(X, \Sigma)$ such that $\succsim_c |_{E_\gamma}$ is cyclic, and hence $\mathcal{W}(X, \Sigma) \neq \mathcal{G}(X, \Sigma)$. Thus, by contraposition, $\mathcal{W}(X, \Sigma) = \mathcal{G}(X, \Sigma)$ implies the well-coveredness of Σ. $\square$

[20] The loop γ, viewed as a loop in the subgraph $\Gamma(X, \Sigma|_{\mathcal{B}_\gamma})$, is what is sometimes referred to as 'chordless' in graph theory.

Appendix II: Combinatorics of Simple Sub-Domains

In this section we take a brief technical detour to prove some results characterizing simple sub-domains and their combinatorics. Our main result is a 'reduction' theorem which may be interpreted as stating that if a loop is contained in some simple sub-domain, then it is contained in some minimal such one, whose 1-skeleton consists exclusively of the loop of interest, and bisections of it. We also obtain a result stating if two simple sub-domains intersect only on along an 'edge,' then the union of the sub-domains is itself simple as well.

Definitions

We begin by recalling some definitions from the theory of simplicial complexes. A **simplicial complex** is a set of vertices $\{v_i\}_{i\in\mathcal{I}}$, and collection of non-empty finite subsets $\{s_j\}_{j\in\mathcal{J}}$ of $\{v_i\}$ called **simplices** such that:

1. Any set consisting of exactly one vertex is a simplex; and

2. Any non-empty subset of a simplex is a simplex.[21]

A simplex of cardinality $(n+1)$ is said to be of dimension n. A simplicial complex is said to be **homogeneously n-dimensional** if every simplex is contained in some n-dimensional simplex.[22] An **n-dimensional pseudomanifold** (with boundary) is a simplicial complex K such that:[23]

(PM.1) K is homogeneously n-dimensional;

(PM.2) Every $(n-1)$-simplex of K is a proper face of at most two n-simplices of K; and

(PM.3) If s and s' are n-simplices of K, then there is a finite sequence $s = s_1, s_2, \ldots, s_k, s_{k+1} = s'$ of distinct n-simplices of K such that for all $1 \leq l \leq k$, s_l and s_{l+1} have an $(n-1)$-face in common.

[21] See Spanier (1989) Section 3.1 (p. 108) for basic definitions.

[22] In some places in the literature, such a simplicial complex is referred to as a **pure** n-dimensional complex.

[23] See Spanier (1989) Section 3.1.C (p. 150) for basic definitions. Note the definition given below allows for boundary; some other sources will require a pseudomanifold to have no boundary, by requiring in condition (PM.2) that 'at most' becomes 'exactly.'

In all that follows, we will fix the assumption that any pseudomanifold is *finite* (in the sense of being composed of finitely many simplices). This will be without of loss of generality for our purposes.

The **boundary** of an n-dimensional pseudomanifold K, denoted $\dot{K}$, is the sub-complex of K consisting of those $(n-1)$-simplices which are the faces of exactly one n-simplex in K. The n-**skeleton** of a complex K, denoted $K^{(n)}$ consists of the sub-complex consisting of all simplices of K of dimension n or lower. Finally, if in addition an n-pseudomanifold K has the property that for all n-simplices s, s' there exists a *unique* path of the form in condition (PM.3), then we will say K is **combinatorially trivial**.

Technical Results

The following result is immediate from the above definitions.

Lemma. *Let (X, Σ) be a choice problem. Then the domain $\mathcal{D}(X, \Sigma)$ is a simplicial complex,*

$$\mathcal{D}(X, \Sigma)^{(0)} = X,$$

$$\mathcal{D}(X, \Sigma)^{(1)} = E_\Gamma,$$

and

$$\mathcal{D}(X, \Sigma)^{(2)} = T_\Gamma.$$

In particular, any simple sub-domain is a sub-complex of the domain that is a combinatorially trivial pseudo-manifold with vanishing 1-dimensional simplicial homology (in $\mathbb{R}$ coefficients). Henceforth we will suppose a fixed underlying choice problem and suppress the arguments (X, Σ).

We now provide purely combinatorial characterization of the simple sub-domains of any domain. It says that if a loop l is contained in some simple sub-domain, then it is contained in a smallest such sub-domain, which is characterized by the properties of having a vertex set equal to the vertex set of l, and the property that l is precisely the boundary of the sub-domain. In particular, this implies that the edge set of this minimal simple sub-domain contains only edges that are 'bisections' of the loop l.

Theorem (Fundamental Theorem of Simple Sub-domains)**.** *Let $\mathcal{D}$ be an arbitrary domain, and $l \subseteq \mathcal{D}^{(1)}$ a loop. There exists a simple sub-domain for l if and only if there exists a simple sub-domain $\mathcal{D}|_{\bar{T}}$ for l that satisfies:*

(i) **Boundary***: $\dot{\mathcal{D}}|_{\bar{T}}^{(1)} = l^{(1)}$; and*

(ii) **Minimality***: The vertex set of $\mathcal{D}|_{\bar{T}}$ equals that of l:*

$$\mathcal{D}|_{\bar{T}}^{(0)} = l^{(0)}.$$

Proof. $(\Longleftarrow)$: Trivial.

$(\Longrightarrow)$: Let $\tilde{T}$ generate a simple sub-domain for l, and consider a chain $\lambda \in C_1(\mathcal{D}|_{\tilde{T}}, \mathbb{R})$, given by:

$$\lambda = \sum_{\sigma \in l} n_\sigma \sigma,$$

with (i) zero coefficients on any $\sigma \in \mathcal{D}|_{\tilde{T}}$ that does not belong to l, and (ii) and such that, for all $\sigma \in l$, the coefficients satisfy $|n_\sigma| = 1$, where signs are chosen so $\lambda \in \ker \partial_1$.[24] As, by topological triviality, $H_1(\mathcal{D}|_{\tilde{T}}, \mathbb{R}) = 0$, $\operatorname{Im} \partial_2 = \ker \partial_1$ hence there exists some chain Λ in $C_2(\mathcal{D}|_{\tilde{T}}, \mathbb{R})$ solving:

$$\partial_2 \underbrace{\Bigg[\sum_{\tau \in \tilde{T}} n_\tau \tau \Bigg]}_{\Lambda} = \lambda$$

with some n_τ possibly equal to zero. Let $\bar{T} = \{\tau \in \tilde{T} : |n_\tau| \neq 0\}$ denote the support of Λ, and suppose for some 2-simplex $\tau \in \mathcal{D}|_{\bar{T}}$ there is a 1-face $\hat{\sigma}$ of τ such that $\hat{\sigma} \in \dot{\mathcal{D}}|_{\bar{T}}$ but $\hat{\sigma} \notin l$. Then we immediately obtain a contradiction: it must be the case actually $n_\tau = 0$, since $\partial_2 \Lambda$ would have coefficient equal in absolute value to $|n_\tau|$ on $\hat{\sigma}$, as by definition of a boundary, τ is the only 2-simplex in $\mathcal{D}|_{\bar{T}}$ containing $\hat{\sigma}$, and $n_{\hat{\sigma}} = 0$ as $\hat{\sigma} \notin l$. Therefore we conclude Λ is supported on a finite sub-collection $\bar{T}$ with the property that $\dot{\mathcal{D}}|_{\bar{T}} \subseteq l \subseteq \mathcal{D}|_{\bar{T}}$.

[24] This is possible as l is a 1-pseudomanifold without boundary; the apparent indeterminacy of the signs of the coefficients in λ is simply a consequence of our being ambivalent about the choice basis for $C_1(\mathcal{D}|_{\tilde{T}}, \mathbb{R})$.

We claim first that $\mathcal{D}|_{\bar{T}}$ is a 2-pseudomanifold. Clearly it satisfies (PM.1) and (PM.2). Suppose for sake of contradiction, that it fails to satisfy (PM.3). Then, by finiteness, there exist a partition of $\bar{T}$ into maximal, non-empty collections of 2-faces $\bar{T}_1, \ldots, \bar{T}_K$, $K > 1$, such that for each k, $\mathcal{D}|_{\bar{T}_k}$ satisfies (PM.3). It follows then that each k, $\mathcal{D}|_{\bar{T}_k}$ is combinatorially trivial.[25] This in turn implies that for all k, $\dot{\mathcal{D}}|_{\bar{T}_k} \neq \varnothing$, as any 'leaf' 2-face contains at least two 1-faces unique to it. Fix an arbitrary k and let $\hat{\Lambda}$ be a 2-chain in $C_2(\mathcal{D}|_{\bar{T}_k}, \mathbb{R})$ whose coefficients are are all 1 in absolute value, with signs chosen so that $\partial_2\hat{\Lambda}$ vanishes on any 1-face not in $\dot{\mathcal{D}}|_{\bar{T}_k}$.[26] Then by construction,

$$\partial_2\hat{\Lambda} = \sum_{\sigma \in \dot{\mathcal{D}}|_{\bar{T}_k}} \hat{n}_\sigma \sigma$$

and for all such σ, $|\hat{n}_\sigma| = 1$. By identity, $(\partial_1 \circ \partial_2)(\hat{\Lambda}) = 0$, and thus for each vertex $x \in \dot{\mathcal{D}}|^{(0)}_{\bar{T}_k} \neq \varnothing$, x is contained in an even number of 1-faces in $\dot{\mathcal{D}}|_{\bar{T}_k}$, and hence $\dot{\mathcal{D}}|_{\bar{T}_k}$ consists of a union of loops. But as $\dot{\mathcal{D}}|_{\bar{T}_{\hat{k}}}$ is a strict subcomplex of $\dot{\mathcal{D}}|_{\bar{T}}$ (lest two partition elements be able to be merged, contradicting their maximality) and hence of l, we obtain a contradiction, as no proper subcomplex of a loop may be a loop. Thus $\mathcal{D}|_{\bar{T}}$ is a pseudomanifold and hence is itself combinatorially trivial as well.

We now verify that $\dot{\mathcal{D}}|_{\bar{T}} = l$. Recall we have already obtained that $\dot{\mathcal{D}}|_{\bar{T}} \subseteq l \subseteq \mathcal{D}|_{\bar{T}}$. Suppose then, for sake of contradiction, that there exists some 1-face $\sigma \in l$, such that $\sigma \notin \dot{\mathcal{D}}|_{\bar{T}}$. Let $\tau \in \mathcal{D}|_{\bar{T}}$ denote one of the two 2-faces of $\mathcal{D}|_{\bar{T}}$ that contains σ, and let K denote the sub-complex of $\mathcal{D}|_{\bar{T}}$ generated by those 2-faces of $\mathcal{D}|_{\bar{T}}$ that may be reached from τ by a sequence of 2-simplices (in the sense of (PM.3)) whose intersections do not contain σ. By construction K satisfies (PM.1) - (PM.3) and is combinatorially trivial; by an argument analogous to that of the preceding paragraph, $\dot{K}$ is a non-empty union of loops.

[25] As $\mathcal{D}|_{\tilde{T}}$ is a combinatorially trivial pseudomanifold by hypothesis, for any k and any pair of 2-faces $T, T' \in \bar{T}_k$ if there were multiple connecting sequences of distinct 2-faces (and by construction there exists at least one such sequence) then there would exist multiple sequences in $\mathcal{D}|_{\tilde{T}}$, as:

$$\mathcal{D}_{\bar{T}_k} \subseteq \mathcal{D}|_{\bar{T}} \subseteq \mathcal{D}|_{\tilde{T}},$$

which cannot be.

[26] This is straightforward due to $\mathcal{D}|_{\bar{T}_k}$ being a combinatorially trivial pseudomanifold.

But $\dot{K} \subsetneq \dot{\mathcal{D}}|_{\bar{T}} \cup \{\sigma\} \subseteq l$, where the first strict inclusion follows from the fact that the complement of K in $\mathcal{D}|_{\bar{T}}$ is a non-empty combinatorially trivial pseudomanifold too. Hence we obtain a contradiction, again because l cannot contain any proper sub-complex that is also a loop, and thus $\dot{\mathcal{D}}|_{\bar{T}} = l$ as claimed.

We turn to verifying our minimality claim, that the vertex sets of $\mathcal{D}|_{\bar{T}}$ and l coincide:

$$\mathcal{D}|_{\bar{T}}^{(0)} = l^{(0)}.$$

Let G denote the undirected graph whose vertex set is given by the 2-faces of $\mathcal{D}|_{\bar{T}}$ and whose edge set determined by the relation of having an intersection containing a 1-face. By combinatorial triviality of $\mathcal{D}|_{\bar{T}}$, G is a tree. Now, suppose toward a contradiction that the vertex sets of $\mathcal{D}|_{\bar{T}}$ and l do not coincide. Since $\dot{\mathcal{D}}|_{\bar{T}} = l$, this implies there is some vertex x of $\mathcal{D}|_{\bar{T}}$ not in l. Now, as $\mathcal{D}|_{\bar{T}}$ is a pseudomanifold, every 1-face σ of $\mathcal{D}|_{\bar{T}}$ that contains x is contained in precisely two 2-simplices. Let $\tilde{G}$ be the subgraph of G consisting of those 2-faces containing x as a vertex. Since each vertex τ of $\tilde{G}$ contains precisely two 1-faces that contain x, by finiteness $\tilde{G}$ is a cycle graph, contradicting the fact that G is a tree (i.e. that $\mathcal{D}|_{\bar{T}}$ is combinatorially trivial). Hence the vertex sets of $\mathcal{D}|_{\bar{T}}$ and l coincide.

Finally, we show the dimension-1 simplicial homology of $\mathcal{D}|_{\bar{T}}$ is zero in real coefficients, our last outstanding claim. As $\mathcal{D}|_{\bar{T}}$ is a 2-pseudomanifold, its collection of 1-faces may be partitioned into two subsets: those faces in $\dot{\mathcal{D}}|_{\bar{T}}$ and those not. By definition, the edge-set of the graph G introduced in the preceding paragraph is in one-to-one correspondence with the the set of 1-faces of $\mathcal{D}|_{\bar{T}}$ not in $\dot{\mathcal{D}}|_{\bar{T}}$. By combinatorial triviality, G is a tree and hence has one more vertex (2-simplex of $\mathcal{D}|_{\bar{T}}$) than edge (1-face of $\mathcal{D}|_{\bar{T}}$ not in $\dot{\mathcal{D}}|_{\bar{T}}$). Similarly, $l = \dot{\mathcal{D}}|_{\bar{T}}$ is a loop, so the number of 1-faces must be the same as the number of vertices of l, which we have established is also the vertex set of $\mathcal{D}|_{\bar{T}}$. The Euler-Poincaré theorem (Munkres (1984) Theorem 22.2) asserts the equivalence of the following two definitions of the Euler characteristic of $\mathcal{D}|_{\bar{T}}$:

$$\chi(\mathcal{D}|_{\bar{T}}) = \dim H_0(\mathcal{D}|_{\bar{T}}, \mathbb{R}) - \dim H_1(\mathcal{D}|_{\bar{T}}, \mathbb{R}) + \dim H_2(\mathcal{D}|_{\bar{T}}, \mathbb{R}) = V - E + F,$$

where V is the number of 0-simplices, E the number of 1-simplices, and F the number of

2-simplices in $\mathcal{D}|_{\bar{T}}$. By the above counting argument for the set of 1-faces of $\mathcal{D}|_{\bar{T}}$, we know:

$$E = \underbrace{V}_{\text{1-faces in } \dot{\mathcal{D}}|_{\bar{T}}.} + \underbrace{F-1}_{\text{1-faces in } \mathcal{D}|_{\bar{T}} \setminus \dot{\mathcal{D}}|_{\bar{T}}.} \tag{1}$$

and hence $\chi(\mathcal{D}|_{\bar{T}}) = 1$. Now, since every 2-simplex in $\mathcal{D}|_{\bar{T}}$ intersects l, $\mathcal{D}|_{\bar{T}}$ is path-connected and hence $\dim H_0(\mathcal{D}|_{\bar{T}}, \mathbb{R}) = 1$. Moreover, by combinatorial triviality, $H_2(\mathcal{D}|_{\bar{T}}, \mathbb{R}) = 0$.[27] Then by Euler-Poincaré, $\dim H_1(\mathcal{D}|_{\bar{T}}, \mathbb{R}) = 0$, and thus $\mathcal{D}|_{\bar{T}}$ is a simple sub-domain as claimed. □

Lemma (Union Lemma)**.** *Let $\mathcal{D}|_T, \mathcal{D}|_{T'}$ be two simple sub-domains whose intersection consists of a single 1-face σ. Then $\mathcal{D}|_T \cup \mathcal{D}|_{T'}$ is a simple sub-domain.*

Proof. As $\mathcal{D}|_T$ and $\mathcal{D}|_{T'}$ are combinatorially trivial, it is immediate that so too is $\mathcal{D}|_T \cup \mathcal{D}|_{T'}$. Then, by the reduced simplicial Mayer-Vietoris theorem (Munkres (1984) Theorem 25.1) there exists an exact sequence:

$$0 \to \tilde{H}_1(\mathcal{D}|_T \cap \mathcal{D}|_{T'}, \mathbb{R}) \to \tilde{H}_1(\mathcal{D}|_T, \mathbb{R}) \oplus \tilde{H}_1(\mathcal{D}|_{T'}, \mathbb{R}) \to \tilde{H}_1(\mathcal{D}|_T \cup \mathcal{D}|_{T'}, \mathbb{R}) \to \tilde{H}_0(\mathcal{D}|_T \cap \mathcal{D}|_{T'}, \mathbb{R})$$

which, making use of topological triviality of $\mathcal{D}|_T$ and $\mathcal{D}|_{T'}$ and the contractibility of $\mathcal{D}|_T \cap \mathcal{D}|_{T'}$ (i.e. $= \sigma$), reduces to:

$$0 \to (0) \to (0) \oplus (0) \to \tilde{H}_1(\mathcal{D}|_T \cup \mathcal{D}|_{T'}, \mathbb{R}) \to 0$$

and hence $\tilde{H}_1(\mathcal{D}|_T \cup \mathcal{D}|_{T'}, \mathbb{R}) = 0$, and equivalently $H_1(\mathcal{D}|_T \cup \mathcal{D}|_{T'}, \mathbb{R})$ by definition of reduced simplicial homology. □

[27] Since $\mathcal{D}|_{\bar{T}}$ is homengeously 2-dimensional it contains no simplices of dimension greater than two, hence $H_2(\mathcal{D}|_{\bar{T}}, \mathbb{R}) = 0$ if and only if the only solution to:

$$\partial_2 \Big[\sum_{\tau \in \mathcal{D}|_{\bar{T}}^{(2)}} \tilde{n}_\tau \tau \Big] = 0$$

is for $\tilde{n}_\tau = 0$ for all $\tau \in \mathcal{D}|_{\bar{T}}^{(2)}$. Clearly for any solution, any $\tau \in \mathcal{D}|_{\bar{T}}$ which contains a 1-face in $\dot{\mathcal{D}}|_{\bar{T}}$ must have have $\tilde{n}_\tau = 0$ by (PM.2). Hence in any non-zero solution to the above, the sub-collection of 2-simplices in $\mathcal{D}|_{\bar{T}}$ with non-zero coefficients must have the property that all of their 1-faces are contained also in some other (hence unique other) member of the sub-collection. But this sub-collection defines a subgraph of the graph G, and the above property implies again that this subgraph can have no leaves, contradicting the fact G is a tree, as $\mathcal{D}|_{\bar{T}}$ is combinatorially trivial. Hence $H_2(\mathcal{D}|_{\bar{T}}, \mathbb{R}) = 0$.

Appendix III: Integrability Proofs

In this section we will address the discrete, ordinal theory of integrability laid out in Section 4. Our methodology is to first interpret the domain associated with a given choice problem as a combinatorial object, a simplicial complex. This object will serve as the domain of a discrete 'total differential equation' that will eventually be built to capture the specifics of a given choice correspondence's revealed preference. In particular, as is the case in the classical theory, the *topology* of the domain of a differential equation reflects the ability of local solutions to be extended into global ones (for example, see Thomas (1934); for discussion of this interpretation in the combinatorial context, see Jiang et al. (2011)).

The Discrete Exterior Calculus

Given a simplicial complex K, a (discrete) n-**form** is a linear functional acting on oriented pieces of the n-dimensional skeleton $K^{(n)}$.[28] Define the space of all n-forms on K as:

$$C^n(K) = \big\{\phi : K^{(n)} \to \mathbb{R} : \phi([x_{\sigma(0)}, \dots, x_{\sigma(n)}]) = \text{sign}(\sigma)\phi([x_0, \dots, x_n])\big\},$$

where $[x_0, \dots, x_n]$ denotes an oriented n-simplex of K and σ is any permutation of $\{0, \dots, n\}$. In particular, the vector space $C^0(K)$ consists precisely of all real valued functions on the vertices; $C^1(K)$ may be interpreted as the space of all real-valued flows on the 1-skeleton of K, where the permutation condition simply ensures that these flows are directed .

There are natural linear operators on forms corresponding to discrete analogues of the traditional operations of the calculus of several variables. Given a 0-form $f \in C^0(K)$, its **gradient** is the 1-form $\text{grad}(f)$ defined via:

$$\text{grad}(f)\big([x_0, x_1]\big) = f([x_1]) - f([x_0]).$$

[28]We intentionally adopt the analyst's terminology of 'forms' rather than the topologist's 'cochain' to further highlight the parallel to the exterior calculus arguments underpinning the solution to the classical integrability problem. See Jiang et al. (2011) and Grady and Polimeni (2010) for an in-depth discussion of the parallels between the smooth and discrete theories.

For a 1-form $F \in C^1(K)$, the **rotation** (or 'curl') of F, denoted $\mathrm{rot}(F)$, is the 2-form defined via:

$$\mathrm{rot}(F)([x_0, x_1, x_2]) = F([x_0, x_1]) + F([x_1, x_2]) - F([x_0, x_2]).$$

A 1-form F is said to be **exact** (or 'integrable') if there exists an $f \in C^0(K)$ such that $\mathrm{grad}(f) = F$. Similarly, if $\mathrm{rot}(F) = 0$, F is said to be **closed**. An exact 1-form is always closed; this may be succinctly stated as $\mathrm{Im}(\mathrm{grad}) \subseteq \mathrm{Ker}(\mathrm{rot})$. In particular, this implies the quotient vector space $\mathrm{Ker}(\mathrm{rot})/\mathrm{Im}(\mathrm{grad})$ is well-defined. This quotient is denoted $H^1(K, \mathbb{R})$ and is known as the first simplicial cohomology group of K (with $\mathbb{R}$-coefficients); its dimension may be interpreted as a measure of how far the closedness of a 1-form is from guaranteeing its exactness, or integrability.

Results

Henceforth we fix a choice problem (X, Σ), and will suppress the argument (X, Σ) appearing in domains. Let $\succeq$ be a binary relation on X that is locally rational.[29] Let γ be a loop in $\mathcal{D}$, and $\mathcal{D}|_{\tilde{T}}$ a simple sub-domain of $\mathcal{D}$ containing γ. A 1-form $F \in C^1(\mathcal{D}|_{\tilde{T}})$ is a **cardinalization** of $\succeq$ on $\mathcal{D}|_{\tilde{T}}$ if, for all 1-faces of $\mathcal{D}|_{\tilde{T}}$:

$$y \succeq x \implies F([x, y]) \geq 0,$$

and

$$y \succ x \implies F([x, y]) > 0.$$

Lemma. *(Closed Cardinalization Lemma) Let $\succeq$ be locally rational, and let $\mathcal{D}|_{\tilde{T}} \subseteq \mathcal{D}$ be a simple sub-domain. Then there exists a closed cardinalization of $\succeq$ on $\mathcal{D}|_{\tilde{T}}$.*

Proof. Let $\{\tilde{\tau}_1, \ldots, \tilde{\tau}_I\}$ be an enumeration of those 2-simplices of $\mathcal{D}|_{\tilde{T}}$ that each contain at least two distinct 1-faces in $\dot{\mathcal{D}}|_{\tilde{T}}$. Using this, we construct an enumeration of all 2-simplices of K as follows: between each $\tilde{\tau}_i, \tilde{\tau}_{i+1}$ insert the unique (via combinatorial triviality) ordered sequence of 2-simplices in $\mathcal{D}|_{\tilde{T}}$ connecting them, omitting any 2-simplices of $\mathcal{D}|_{\tilde{T}}$ that have appeared in the construction prior. Let $\{\tau_1, \ldots, \tau_J\}$ denote this enumeration, and let $\mathcal{D}|_{\tilde{T}}^j$ denote the sub-complex of $\mathcal{D}|_{\tilde{T}}$ generated by the first j elements of this enumeration.

[29] That is, for all $T \in T_\Gamma$, $\succeq|_T$ is complete and transitive.

We now inductively construct our closed cardinalization of $\succeq$. First, note that there is trivially a closed cardinalization of $\succeq$ on $\mathcal{D}|^1_{\tilde{T}}$: $\succeq$ restricted to the vertices of τ_1 is complete and transitive by local rationality, hence admits a utility function u_1 on these vertices. Let $F_1 \in C^1(\mathcal{D}|^1_{\tilde{T}})$ be defined as $\text{grad}(u_1)$. For our inductive step, suppose now that there is a closed cardinalization $F_j \in C^1(\mathcal{D}|^j_{\tilde{T}})$ of $\succeq$ on $\mathcal{D}|^j_{\tilde{T}}$, for some $j < J$. By analogous logic, there is a utility function u_{j+1} representing $\succeq$ restricted to the vertices of τ_{j+1}. Let $\tilde{F}_{j+1} = \text{grad}(u_{j+1})$ be the closed 1-form on the the complex generated by τ_{j+1} alone. By virtue of the structure of the enumeration constructed above, τ_{j+1} and τ_j intersect on exactly a single 1-face, σ with vertex set $\{a, b\}$. There exists some $c \in \mathbb{R}_{++}$ such that $F_j([a, b]) = c\tilde{F}_{j+1}([a, b])$, with c unique if $\succeq$ is strict over this pair. Then define:

$$F_{j+1}([x, y]) = \begin{cases} F_j([x, y]) & \text{if } [x, y] \not\subset \tau_{j+1} \\ c\tilde{F}_{j+1}([x, y]) & \text{if } [x, y] \subset \tau_{j+1}, \end{cases}$$

completing the proof. $\square$

Theorem (Ordinal Integrability Theorem)**.** *Let (X, Σ) be a choice problem with $\mathcal{D}(X, \Sigma)$ a simple domain. Then a choice correspondence $c \in \mathcal{C}(X, \Sigma)$ is strongly rationalizable if and only if:*

(i) It obeys the weak axiom; and

(ii) It is locally rationalizable.

Moreover, (i) and (ii) are jointly equivalent to the strong rationalizability of c if and only if $\mathcal{D}(X, \Sigma)$ is simple.

Proof. We begin first by verifying (i) and (ii) are equivalent to strong rationalizability for simple $\mathcal{D}$. Clearly, strong rationalizability always implies (i) and (ii), regardless of the structure of $\mathcal{D}$: any rationalizing weak order $\succeq_c$ of course is a local rationalization and implies $(\succsim_c, \succ_c)$ obeys the weak axiom.

Now, suppose $\mathcal{D}$ is simple, and let $c \in \mathcal{W}(X, \Sigma)$ be locally rationalizable. Let $\gamma \subseteq \mathcal{D}$ be an arbitrary loop. We will show that $\succsim_c|_{E_\gamma}$ cannot be cyclic. As γ is a loop, by simplicity of $\mathcal{D}$ there exists a simple sub-domain $\mathcal{D}|_{\tilde{T}} \subseteq \mathcal{D}$ containing γ, and $\succeq$ a local rationalization of $\succsim_c$ on $\mathcal{D}|_{\tilde{T}}$. By the preceding lemma there exists a closed cardinalization of $\succeq$ on $\mathcal{D}|_{\tilde{T}}$, which we will denote by $F \in C^1(\mathcal{D}|_{\tilde{T}})$. By the cohomology universal coefficient theorem (see Munkres (1984) Theorem 53.1), there exists an isomorphism between $H_1(\mathcal{D}|_{\tilde{T}}, \mathbb{R})$ and $H^1(\mathcal{D}|_{\tilde{T}}, \mathbb{R})$ (see Munkres (1984) Corollary 53.6 or Jiang et al. (2011) Theorem 4), and hence as $\mathcal{D}|_{\tilde{T}}$ is topologically trivial, $H^1(\mathcal{D}|_{\tilde{T}}, \mathbb{R}) = 0$, and therefore there exists an $f \in C^0(\mathcal{D}|_{\tilde{T}})$ such that:

$$\mathrm{grad}(f) = F.$$

Define the binary relation $\geq^*$ on the vertex set $\mathcal{D}|_{\tilde{T}}^{(0)}$ via $x_0 \geq^* x_1 \iff f(x_0) \geq f(x_1)$ (resp. strict). This is a weak order on the vertices of $\mathcal{D}|_{\tilde{T}}$ which, by consistency of F, is an extension of $\succeq$ on $\mathcal{D}|_{\tilde{T}}$.[30] Thus $\succsim_c|_{E_\gamma}$ is acyclic. As γ was arbitrary, and every potential cycle of $\succsim_c$ must be supported on some loop in $\mathcal{D}$, each contained in some simple sub-domain, we conclude $\succsim_c$ is acyclic. Thus, for all $c \in \mathcal{W}(X, \Sigma)$, if c is also locally rationalizable, it must satisfy the generalized axiom and hence is strongly rationalizable.

We now show that if $\mathcal{D}$ is not simple, (i) and (ii) do not imply strong rationalizability. Suppose, then, that $\mathcal{D}$ is not simple. Then, as there exists a loop contained in no simple sub-domain, there exists a shortest such loop, which we will denote γ, with $E_\gamma = \big\{\{x_0, x_1\}, \ldots, \{x_{n-1}, x_n\}, \{x_n, x_0\}\big\}$. We know $|V_\gamma|$ must be strictly greater than three, lest V_γ be a triple in $\mathcal{D}$ and hence this triple serve trivially as a simple sub-domain containing γ. We now prove that for all $e \in E_\Gamma$, $e \subseteq V_\gamma$ if and only if $e \in E_\gamma$, that is, that γ is a chordless loop in $\Gamma(X, \Sigma)$. Clearly $E_\gamma \subseteq E_\Gamma$. Thus, for sake of contradiction, suppose there exists an $e \in E_\Gamma$ with $e \subseteq V_\gamma$ but $e \notin E_\gamma$. Then without loss of generality, $e = \{x_j, x_k\}$ with $k > j + 1$. Hence we obtain two loops, γ_1 and γ_2 via:

$$E_{\gamma_1} = \big\{\{x_0, x_1\}, \ldots, \{x_{j-1}, x_j\}, \{x_j, x_k\}, \{x_k, x_{k+1}\}, \ldots, \{x_n, x_0\}\big\}$$

and

$$E_{\gamma_2} = \big\{\{x_j, x_{j+1}\}, \ldots, \{x_{k-1}, x_k\}, \{x_k, x_j\}\big\},$$

[30] It is generally an extension of $\succeq$ (which itself extends the revealed preference $\succsim_c$) as $\geq^*$ is complete and thus generally relates vertices not connected by any edge in $\mathcal{D}|_{\tilde{T}}^{(1)}$.

both shorter than γ. By the minimality of γ, there exist simple sub-domains $\mathcal{D}|_{\tilde{T}_1}$ and $\mathcal{D}|_{\tilde{T}_2}$ of $\mathcal{D}$ for γ_1 and γ_2 respectively, and by the fundamental theorem of simple sub-domains, these complexes may be taken to intersect only on the 1-face $\{x_j, x_k\}$. But by the union lemma, $\mathcal{D}|_{\tilde{T}_1} \cup \mathcal{D}|_{\tilde{T}_2}$ is a simple sub-domain for γ, a contradiction. Thus we conclude that for all $e \in E_\Gamma$, $e \subseteq V_\gamma$ if and only if $e \in E_\gamma$.

We have shown that $\gamma \subseteq \Gamma(X, \Sigma)$ is a chordless loop in the budget graph. Thus, there exists a cyclic collection for γ, denoted $\mathcal{B}_\gamma = \{B_1, \ldots, B_n\} \subseteq \Sigma$ such that for all $0 \leq j \leq n$:

$$\{x_j, x_{j+1}\} \subseteq B_j,$$

and that this collection is uncovered: as $|V_\gamma| > 3$ and γ is chordless, no budget in *all of* Σ contains any pair of non-adjacent points in γ. For all $B \in \Sigma|_{\mathcal{B}_\gamma}$, let:

$$\tilde{c}(B) = \begin{cases} e_j \cap e_{j+1} & \text{if } \exists\, e_j \text{ s.t. } e_j = B \cap V_\gamma \\ B \cap V_\gamma & \text{if } |B \cap V_\gamma| = 1 \\ B & \text{else,} \end{cases}$$

and for all $B \in \Sigma$ define:

$$c(B) = \begin{cases} \tilde{c}(B) & \text{if } B \in \Sigma|_{\mathcal{B}_\gamma} \\ B \setminus (\cup_{\tilde{B} \in \mathcal{B}_\gamma} \tilde{B}) & \text{else.} \end{cases}$$

By an argument analogous to that in the proof of Theorem 1, $c \in \mathcal{W}(X, \Sigma)$ and not $\mathcal{G}(X, \Sigma)$.

We now verify that c is nonetheless locally rationalizable. To do this, we will explicitly construct a local rationalization $\succeq$. First, for all $e \in E_\gamma$, let:

$$x_i \prec x_{i+1}.$$

Thus for all pairs $\{x, y\} \in E_\gamma$, $x \succ y$ if and only if $x \succ_c y$. For all $e \in E_\Gamma \setminus E_\gamma$ that intersect V_γ, we have shown this intersection must be singleton. For all such e, we know e is of the form $\{a, x_i\}$ for some $x_i \in V_\gamma$. For all pairs $\{a, x_i\}$ with $a \in (\cup_{\tilde{B} \in \mathcal{B}_\gamma} \tilde{B})$, define:

$$a \prec x_i,$$

and if $a \notin (\cup_{\tilde{B}\in\mathcal{B}_\gamma}\tilde{B})$, let:

$$a \succ x_i.$$

Finally, for those pairs $\{a, b\}$ that do not intersect V_γ, we consider two cases. If, either $\{a, b\} \subseteq (\cup_{\tilde{B}\in\mathcal{B}_\gamma}\tilde{B})$ or $\{a, b\} \subseteq X \setminus (\cup_{\tilde{B}\in\mathcal{B}_\gamma}\tilde{B})$, then let $a \succeq b$ and $b \succeq a$. If exactly exactly one element (without loss a) of $\{a, b\}$ is contained in $(\cup_{\tilde{B}\in\mathcal{B}_\gamma}\tilde{B})$, then let $b \succ a$. Finally, let $\succeq^*$ denote the reflexive closure of $\succeq$. Then $\succeq^* \supseteq \succsim_c$, and, by construction, $\succeq^*$ is locally rational. This follows from (i) for all $\{a, b\} \in E_\Gamma$, either $a \succeq^* b$ or $b \succeq^* a$, and (ii) for every $T \in T_\Gamma$, T only contains at most one pair in γ, as γ is chordless and of length greater than three. Thus, in particular, if T contains an edge of γ, denoted $\{x, y\}$, it contains some element z such that either $z \succ^* x, y$ or $x, y \succ^* z$. If T contains no edges of γ, then $\succeq^* |_T$ is clearly complete and transitive, and hence $\succeq^*$ is locally rational. $\square$

Proof of Theorems 3 and 4

Theorem. *Let (X, Σ) be a choice problem. Then Σ is well-covered if and only if both:*

(i) $\mathcal{D}(X, \Sigma)$ is simple; and

(ii) If $c \in \mathcal{W}(X, \Sigma)$ then c is locally rationalizable.

Proof. Immediate from Theorems 1 and 2. $\square$

Proposition. *Let (X, Σ) be a cardinality-constrained choice problem. Then every $c \in \mathcal{W}(X, \Sigma)$ is locally rationalizable if and only if:*

$$\mathcal{T}_\Gamma = \Sigma_3.$$

Proof. ($\Longleftarrow$): If $T_\Gamma = \Sigma_3$, consider the revealed preference of any choice function c obeying the weak axiom. If, for any pair $\{x, y\}$ contained in some $T \in \mathcal{T}_\Gamma$, we have neither $x \succsim_c y$ nor $y \succsim_c x$, it means that for every $T' \in \mathcal{T}_\Gamma = \Sigma_3$, it is the case that that neither x nor y are chosen, hence $c(T') = T' \setminus \{x, y\}$. Note that, for any $T \in \Sigma_3$, at most one pair of elements may not have any preference revealed between them, as T is a budget itself so some choice must occur on it. Thus adding both (x, y) and (y, x) to $\succsim_c$ for every such $\succsim_c$-unrelated pair $\{x, y\}$ yields a locally rational extension.

($\Longrightarrow$): We proceed by contraposition. Suppose $\mathcal{T}_\Gamma \neq \Sigma_3$. Of course $\Sigma_3 \subseteq \mathcal{T}_\Gamma$, hence there exists some $\{x, y, z\} \in \mathcal{T}_\Gamma$ that is not a budget itself, but every pair of elements in it is contained in some budget. It is immediate then, due to the cardinality constraints on Σ, that $\{x, y\}, \{y, z\}, \{z, x\}$ is a loop in $\Gamma(X, \Sigma)$ that possesses an uncovered cyclic collection. By Lemma 2 we obtain the existence of a choice correspondence obeying the weak axiom whose revealed preference exhibits a three-cycle on this loop. Since the vertex set of this loop is in $\mathcal{T}_\Gamma$, this choice correspondence cannot be locally rationalizable. $\square$

Theorem. *Let (X, Σ) be a cardinality-constrained choice problem. Then $\mathcal{W}(X, \Sigma) = \mathcal{G}(X, \Sigma)$ if and only if $\mathcal{D}(X, \Sigma)|_{\Sigma_3}$ is simple.*

Proof. ($\Longrightarrow$): Suppose $\mathcal{W}(X, \Sigma) = \mathcal{G}(X, \Sigma)$. Then by Theorem 3, $\mathcal{D}(X, \Sigma)$ is simple and every choice correspondence in $\mathcal{W}(X, \Sigma)$ is locally rationalizable. By the preceding proposition it follows $T_\Gamma = \Sigma_3$, and thus $\mathcal{D}(X, \Sigma)|_{\Sigma_3} = \mathcal{D}(X, \Sigma)|_{T_\Gamma}$ is simple too.

($\Longleftarrow$): Suppose $\mathcal{D}(X, \Sigma)|_{\Sigma_3}$ is simple, and let $\gamma \subseteq \Gamma(X, \Sigma)$ be an arbitrary loop. Then there exists some sub-collection $\tilde{\mathcal{T}}$ of three-good budgets that generate a simple sub-domain containing γ. By the fundamental theorem of simple subdomains, we may take this simple sub-domain's edge set to consist solely of edges of γ and bisections of γ. But, by combinatorial triviality, there exists a 'leaf' triangle in this sub-domain, hence for this triangle there exists a pair of edges $\{x, y\}, \{y, z\} \in E_\gamma$ such that $\{x, y, z\} \in \Sigma_3$. In particular, this implies that every cyclic collection for γ is covered, and by the arbitrariness of γ, Σ is well-covered. Theorem 1 then completes the proof. $\square$

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