How Strong is the Weak Axiom*

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Abstract

We investigate the manner in which the power of the weak axiom of revealed preference is affected by the completeness of the choice environment. We fully characterize those domains on which the weak axiom coincides with strong rationalizability for arbitrary choice correspondences. We also provide a related result that characterizes those domains on which the strong rationalizability of a choice correspondence is equivalent to (i) the satisfaction of the weak axiom, and (ii) the strong rationalizability of its restrictions to suitable collections of small sets. Our proof technique involves a generalization of many of the differential concepts of classical demand theory to the abstract choice model. We conclude with an application to the problem of aggregating incomplete preferences.

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1 Introduction

The weak axiom of revealed preference, corresponding to the absence of pairwise reversals in observed choice behavior, is among the most elementary and normatively appealing consistency criteria. A particularly striking feature of the weak axiom is how dependent its implications are upon the structure of the domain of choice. When choice is observed on a complete collection of budgets, consistency with respect to the weak axiom is equivalent to rational behavior: the weak axiom completely characterizes the testable implications of rationality (see Arrow (1959), Sen (1971)).[1] Conversely, when choice is observed only on an exceedingly sparse collection of budgets, the satisfaction of the weak axiom may become vacuous.

Complete domain hypotheses are commonplace in choice theory. In spite of this, the manner in which the structure of the domain of choice affects the implications of the weak axiom is generally very poorly understood. In the context of experiments, this implies a non-trivial interaction between the experimental design, that is the choice of which budgets to solicit subjects' choices from, and the interpretation of any potential inconsistency. For example, if the weak axiom of revealed preference is characteristic of rationality for a given experiment, then clearly no choice cycle of length three or more can occur in isolation: there must also be a choice reversal. For such experiments, the testable implications of the transitivity of preference are wholly subsumed by pairwise coherency of choices.

Reliance on such assumptions also limit our ability to test new models. It is common to characterize the testable implications of such theories under the assumption of a complete domain. Broadly speaking, the consequence of this assumption is that the empirical content of these models then tends to be characterized by an appropriate

[1] This is sometimes aptly referred to as the 'fundamental theorem of revealed preference.' See, for example, Ok and Tserenjigmid (2019).

variant of the weak axiom (e.g. Manzini and Mariotti (2007), Masatlioglu et al. (2012), Evren et al. (2019)) or at the least to rely heavily on the observation that on a complete domain, 'all cycles imply two-cycles' (e.g. Bernheim and Rangel (2009)). Outside the realm of theory, however, full domain hypotheses are generally difficult to justify on either positive or normative grounds. De Clippel and Rozen (2014) seek to understand what can be empirically tested under incomplete data; our work here may be seen as part of a dual approach of trying to better understand how robust such results are to the relaxation of these assumptions without fundamentally altering their testable implications. It seems likely that future results in this direction will require ideas formally extending those studied here in the 'base case.'

We undertake the systematic study of how the power of the weak axiom varies with the richness of the collection of budgets choice is sampled on. In particular, we fully characterize those choice environments for which the weak axiom of revealed preference exhausts the testable implications of rational choice. We show that the class of environments includes not only complete collections of budgets, but also considerably smaller ones, and the property of having a strong weak axiom is not, in general, preserved under the addition of new budgets nor the restriction to sub-collections. We also consider the related problem, spiritually similar to the integrability theory of classical demand, of when the weak axiom, in conjunction with a 'local' no-cycles condition, characterizes rational choice. It turns out that in general, such a theories also require a suitably rich domain of budgets, though a weaker richness condition than is required for the weak axiom alone to suffice.

Example 1. Consider four alternatives $\{a, b, c, d\}$. Suppose an individual is presented with choices between $\{a, b\}$, $\{b, c\}$, $\{c, d\}$, and $\{d, a\}$. If this individual were to choose a in the presence of b, b in the presence of c and so forth cyclically, her choice behavior would be consistent with the weak axiom. This is because her choice behavior contains no preference *reversal*.[2] However, it would be inconsistent with preference maximization, as it would violate the generalized axiom: it contains a *cycle*.

Suppose now the agent were additionally presented with choices over $\{a, b, c\}$. The presence of the cycle from her other choices would necessarily force her to make another choice cycle over other alternatives: if she did not choose exclusively a as her most-preferred alternative, she would create a revealed preference reversal when this choice was considered alongside those preceding it. But, were she to choose exclusively a, then by revealing a to be preferable to c she would have chosen cyclically over a, d, and c. The structure of this collection of budgets ensures that any cycle of choices over all four alternatives necessarily induces other choice cycles in the data, though not necessarily a choice reversal.

Finally, suppose the agent is now presented with choices over the four binary budgets, $\{a, b, c\}$, and $\{c, d, a\}$. The presence of the cycle from her first four choices now necessarily forces her to make a preference reversal in her choices from the latter two budgets. If the agent were to choose anything but a from $\{a, b, c\}$, a reversal would be immediate. But then *any* choice from $\{a, c, d\}$ constitutes a choice reversal. Though this collection of budgets is far from complete, the budgets nevertheless intersect in such a manner as to force any revealed preference cycle to necessarily induce a concomitant revealed preference reversal. Were these choice sets selected by an experimenter to be presented to the individual, the experimental setup would preclude the existence of testable implications of preference transitivity beyond pairwise coherent choice. ▌

[2] In fact, it would be impossible for the agent to *violate* the weak axiom as no budget in this environment contains a common pair of alternatives.

2 The Ex-Ante Power of the Weak Axiom

2.1 Preliminaries

Let X be an arbitrary set of **alternatives** from which an agent chooses. Let $\Sigma \subseteq 2^X \setminus \{\varnothing\}$ be a collection of **budgets** which we observe the agent choose from. We interpret the collection Σ as capturing the manner in which we are able to sample an agent's choices: we can observe an agent's choice on a set B if and only if it belongs to Σ. When Σ contains all non-empty, finite subsets of X, we will say that Σ is **complete**. We refer to the tuple (X, Σ) as a **choice environment**.

A mapping $c : \Sigma \to 2^X \setminus \{\varnothing\}$ is a **choice correspondence** if, for all $B \in \Sigma$, it satisfies $c(B) \subseteq B$. Let $\mathcal{C}(X, \Sigma)$ denote the collection of all choice correspondences for the environment (X, Σ). Given a choice correspondence $c \in \mathcal{C}(X, \Sigma)$, a preference relation $\succeq$ on X **strongly rationalizes** c if, for every budget we observe choice on, the chosen element(s) are precisely those $\succeq$-maximal alternatives:

$$(\forall B \in \Sigma) \;\; c(B) = \big\{x \in B : \forall y \in B, x \succeq y\big\}.$$

Given a choice correspondence c, its revealed preference is a pair of relations $(\succsim_c, \succ_c)$ defined via: $x \succsim_c y$ if there exists some $B \in \Sigma$ such that $x, y \in B$ and $x \in c(B)$, and $x \succ_c y$ if there exists some $B \in \Sigma$ such that $x, y \in B$, $x \in c(B)$ and $y \notin c(B)$.

A choice correspondence c satisfies the **weak axiom** of revealed preference if it contains no pairwise reversals: $x \succsim_c y$ implies $y \not\succ_c x$.[3] Notably, for choice correspondences satisfying the weak axiom, $\succ_c$ is indeed the asymmetric part of $\succsim_c$, allowing us to speak of a single revealed preference relation for such correspondences. We say c obeys the **generalized axiom** of revealed preference (sometimes referred to as 'congruence') if

[3] Mariotti (2008) provides a characterization of those choice correspondences that obey the weak axiom, for general environments, in terms of the the ability of the choices to be 'justified' by an asymmetric relation.

$(\succsim_c, \succ_c)$ contains no finite cycles of the form:

$$x_0 \succsim_c x_1 \succsim_c \cdots \succsim_c x_N \succ_c x_0,$$

It is without loss to suppose that these alternatives are all distinct, as any cycle containing multiple appearances of the same alternative necessarily also contains a sub-cycle consisting only of distinct alternatives. We will denote the set of all choice correspondences for the environment (X, Σ) that satisfy the weak and generalized axioms, respectively, by $\mathcal{W}(X, \Sigma)$ and $\mathcal{G}(X, \Sigma)$. It was shown by Richter (1966), making use of an extension theorem due to Szpilrajn (1930), that a choice correspondence is strongly rationalizable by a preference relation if and only if it obeys the generalized axiom.[4] In light of this, we will interchangeably refer to the satisfaction of the generalized axiom as strong rationalizability.

2.2 A Characterization

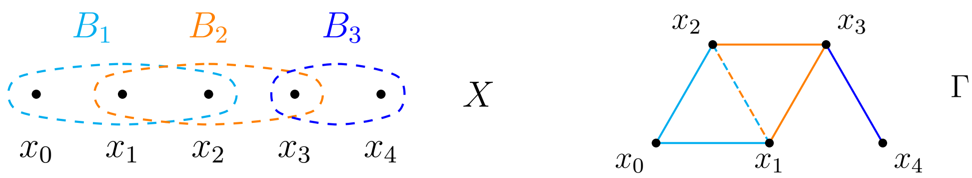
For purposes of combinatorial bookkeeping, it will be helpful to define an auxiliary structure that, for a given choice environment (X, Σ), encodes precisely which pairs of alternatives it is even possible for a preference to be revealed between. Let $\Gamma(X, \Sigma)$ be an undirected graph whose vertex set is X, and whose edge-set E_Γ is given by the (symmetric) relation of two vertices belonging to some common budget:

$$\{x, y\} = e_{xy} \in E_\Gamma \iff \exists B \in \Sigma \text{ s.t. } \{x, y\} \subseteq B.$$

We term $\Gamma(X, \Sigma)$ the **budget graph**. Equivalently, the budget graph is the smallest undirected network with vertex set X for which the reflexive closure of the edge relation contains every revealed preference arising from a choice correspondence satisfying the weak axiom.[5]

[4] We note, however, that Szpilrajn (1930) acknowledges the priority of Banach, Kuratowski, and Tarski in discovering, though not publishing, the result.

[5] The revealed preference arising from the 'complete indifference' choice correspondence, for example, obtains this bound.



(a) A choice environment with five alternatives and three budget sets.

(b) The budget graph associated with this environment.

Figure 1: A simple choice environment and its corresponding budget graph. The coloring of the edges in the budget graph indicates which budgets are responsible for the edge's inclusion in the graph.

For a given $c \in \mathcal{W}(X, \Sigma)$ and any $e \in E_\Gamma$ there is a well-defined (possibly empty) restriction of the revealed preference $\succsim_c$ to the edge $\succsim_c |_e$. This is because an edge $e = \{x, y\}$ is itself a two-element subset of the graph's vertex set, thus:

$$\succsim_c |_e = \succsim_c \cap \{x, y\} \times \{x, y\}$$

is well-defined. Similarly, given a collection of edges $E' \subseteq E_\Gamma$, we define:

$$\succsim_c |_{E'} = \bigcup_{e \in E'} \succsim_c |_e.$$

A loop in Γ is a connected, finite subgraph $\gamma = (V_\gamma, E_\gamma)$ such that every vertex in V_γ belongs to precisely two edges in E_γ. Given a loop $\gamma \subseteq \Gamma(X, \Sigma)$, a collection of budgets $\mathcal{B}_\gamma \subseteq \Sigma$ is a **cyclic collection** for γ if, for every $e \in E_\gamma$ there exists a $B \in \mathcal{B}_\gamma$ with $e \subseteq B$. A cyclic collection for a loop γ is simply a collection of budgets for which there is some choice correspondence $\tilde{c} \in \mathcal{C}(X, \mathcal{B}_\gamma)$ that reveals a preference on every edge in the loop.[6] Our choice of terminology, however, betrays intent: we will be specifically interested in those collections that allow for cyclic choices around the loop and, in particular, those which admit extensions to all of Σ that obey the weak axiom.

Our ability use a particular cyclic collection to construct a choice correspondence that satisfies the weak axiom, but not the generalized, depends critically on how the

[6]Note that for every loop in $\Gamma(X, \Sigma)$, there exists at least one cyclic collection.

collection intersects the remaining budgets in Σ. Given a loop γ and cyclic collection $\mathcal{B}_\gamma$, we say $\mathcal{B}_\gamma$ is **covered** if either:

(i) There exists a $\bar{B} \in \Sigma|_{\mathcal{B}_\gamma}$ such that $V_\gamma \subseteq \bar{B}$; or

(ii) There exists a $\bar{B} \in \Sigma|_{\mathcal{B}_\gamma}$ such that $\bar{B}$ contains a pair of elements of V_γ that are not connected by any edge in E_γ,

where we define the restricted collection $\Sigma|_{\mathcal{B}_\gamma}$ via:

$$\Sigma|_{\mathcal{B}_\gamma} = \left\{ \bar{B} \in \Sigma : \bar{B} \subseteq \bigcup_{B \in \mathcal{B}_\gamma} B \right\}.$$

Note that condition (i) implies (ii) if and only if $|V_\gamma| > 3$. Practically speaking, covering budgets can be interpreted as choice sub-problems that are severely constrained by choices on a cyclic collection. If choices on some cyclic collection are constitute a GARP violation, it is easy to choose from budgets not contained within the cyclic collection without creating a WARP violation, by simply choosing from the (non-empty) subset of alternatives that do not lie within the cyclic collection. If a budget covers the cyclic collection, however, then the ability of a subject to make a pairwise consistent choice from the covering budget is constrained.

Call a choice environment (X, Σ) **well-covered** if, for every loop γ in the budget graph $\Gamma(X, \Sigma)$, every cyclic collection $\mathcal{B}_\gamma$ for γ is covered. Well coveredness, in essence, generalizes the classical argument that on a complete domain, every GARP violation implies a WARP violation (given a GARP violation, the complete domain forces the subject to choose from precisely the subset of alternatives making up the cycle). It requires instead only that the agent be forced to choose from some budget covering the collection on which the cycle is chosen. It turns out this is enough: the well-coveredness of Σ is both necessary and sufficient for the weak axiom of revealed preference to coincide with strong rationalizability for *any* choice correspondence.

Theorem 1. *Let (X, Σ) be a choice environment. The weak axiom of revealed preference is necessary and sufficient for strong rationalizability if and only if (X, Σ) is well-covered.*

Consider again the example from the introduction. In the case where the agent was presented with four budgets $\{a, b\}$, $\{b, c\}$, $\{c, d\}$, and $\{a, d\}$, the budget graph has a single loop, and the sole cyclic collection for this loop is uncovered. Thus this choice environment is not well-covered, and it is of course possible for the agent to choose cyclically in a manner violating the generalized axiom but consistent with the weak. Now, consider the environment when the budget $\{a, b, c\}$ is added. This new budget serves to cover the loop of length four. However, it also adds two new loops of length three to the budget graph, formed by addition of the bisecting edge $\{a, c\}$. All of the cyclic collections for the loop with edges $\{a, b\}$, $\{b, c\}$, and $\{c, a\}$ are covered. However this is not true for the loop with edges $\{c, d\}$, $\{d, a\}$, $\{a, c\}$. Only by also adding yet another budget, $\{c, d, a\}$, is well-coveredness achieved. This last budget adds no new loops to the budget graph but, critically, serves to ensure that the cyclic collection for the loop $\{c, d\}$, $\{d, a\}$, $\{a, c\}$ becomes covered. It is this interlocking nature of the budget collections in the choice environment that well-coveredness characterizes.

2.3 Proof Sketch

The proof of the necessity of the well-coveredness of (X, Σ) for the weak axiom to coincide with the generalized proceeds by contraposition. We exhibit a means of constructing a choice correspondence, obeying the weak axiom but not the generalized, that relies only on the existence of a single loop with a single uncovered cyclic collection. The interested reader is referred to the Appendix. The proof of sufficiency is split over three lemmas. The first is a simple extension result which says, if we are given a loop γ and cyclic collection for it $\mathcal{B}_\gamma$, that if we can find a choice correspondence $\tilde{c}$ on the restricted domain $\Sigma|_{\mathcal{B}_\gamma}$ that cycles on γ and obeys the weak axiom, then there is no obstruction to extending $\tilde{c}$ to the full domain Σ.

Lemma 1. *Let (X, Σ) be a choice environment and let γ be a loop in $\Gamma(X, \Sigma)$. There exists choice function $c \in \mathcal{W}(X, \Sigma)$ such that $\succsim_c |_{E_\gamma}$ contains a cycle if and only if there exists a cyclic collection $\mathcal{B}_\gamma$ and choice function $\tilde{c} \in \mathcal{W}(X, \Sigma|_{\mathcal{B}_\gamma})$ such that $\succsim_{\tilde{c}} |_{E_\gamma}$ contains a cycle.*

The next lemma characterizes those minimal cycles that can arise from a choice correspondence that satisfies the weak axiom. It says that about any triangle in the budget graph, there is a choice correspondence that both (i) satisfies the weak axiom, and (ii) chooses cyclically around this triangle if and only if there exists an uncovered cyclic collection for the triangle.

Lemma 2. *Let (X, Σ) be a choice environment and let γ be a loop in $\Gamma(X, \Sigma)$ with $|V_\gamma| = 3$. Then there exists a choice correspondence $c \in \mathcal{W}(X, \Sigma)$ with $\succsim_c |_{E_\gamma}$ a cycle if and only if there exists a cyclic collection $\mathcal{B}_\gamma$ that is not covered.*

Unfortunately, such a clean characterization does not obtain for longer loops. Lemma 3 however shows that, for loops of length four or more, if every cyclic collection for the loop is covered, then even if we cannot rule out the existence of a $c \in \mathcal{W}(X, \Sigma)$ that chooses cyclically around the loop, if such a c exists, it induces at least one other cycle elsewhere, around some strictly shorter loop.

Lemma 3. *Let (X, Σ) be a choice environment and let γ be a loop in $\Gamma(X, \Sigma)$ with $|V_\gamma| > 3$. Suppose there exists a choice correspondence $c \in \mathcal{W}(X, \Sigma)$ where $\succsim_c |_{E_\gamma}$ contains a cycle. If every cyclic collection $\mathcal{B}_\gamma$ is covered, then there exists a loop γ' in $\Gamma(X, \Sigma)$ such that $|V'_\gamma| < |V_\gamma|$ and $\succsim_c |_{E_{\gamma'}}$ contains a cycle.*

The 'sufficiency' direction of Theorem 1 then follows from a straightforward contraposition argument: suppose there exists some choice correspondence c which satisfies the weak, but not generalized, axiom. Then c contains some cycle of length three or more around some loop in the budget graph. If the loop was of length three, then by

Lemma 2 the loop contains an uncovered cyclic collection and we conclude (X, Σ) is not well-covered. If the loop was of length four or greater and contains an uncovered cyclic collection, we again conclude (X, Σ) is uncovered, thus suppose that all of its cyclic collections are covered. Then by Lemma 3 there is a shorter cycle as well. Iterating this logic finitely many times, we obtain either a loop of length greater than three with an uncovered cyclic collection, or a cycle of length three, which by Lemma 2 implies an uncovered cyclic collection. In both these cases we conclude that (X, Σ) is not well-covered.

2.3.1 Examples of Well-covered Environments

Firstly, any complete environment is well-covered. Thus Theorem 1 extends the classical results of Arrow (1959) and later Sen (1971).

Example 2 (Complete Abstract Environments)**.** Let X be a set, and suppose Σ contains all finite subsets of X. Then Σ is well-covered: letting γ be a loop, $V_\gamma \in \Sigma$. More generally, it is straightforward to show that if Σ either contains all cardinality two or all cardinality three budgets, it is well-covered.

More generally, if the budget collection is closed under finite unions, then it is well-covered. See, for example, Theorem 4 in Kochov (2010).

Example 3 (Collections Closed Under Unions)**.** Let X be a set and suppose that, for all $B, B' \in \Sigma$, that $B \cup B' \in \Sigma$. Then Σ is well covered: for any loop γ, let $\mathcal{B}_\gamma$ denote an arbitrary cyclic collection. Since E_γ is finite, there exists a finite sub-collection of $\mathcal{B}_\gamma$ that is also a cyclic collection for γ. The union of this sub-collection is a budget by hypothesis, and covers $\mathcal{B}_\gamma$.

Another example of well-covered budget collections arise when there is some natural (weak) order on the space of alternatives, and budgets consist of intervals in this order. Such environments naturally arise when choice sets are defined simply by upper and lower bounds.

Example 4 (Interval Budgets)**.** Suppose $(X, \leq_X)$ is a weakly ordered set, and that Σ consists of order intervals, i.e. sets of the form $[x, y] = \{z \in X : x \leq_X z \leq_X y\}$, then it is well-covered. Letting γ be a loop in the budget graph, since V_γ is finite, it contains a $\leq_X$-minimal element, x_i. Without loss, suppose the adjacent vertices satisfy: $x_{i-1} \leq_X x_{i+1}$. Then, since budgets are intervals, every budget for the edge $\{x_i, x_{i+1}\}$ contains x_{i-1}, implying every cyclic collection for γ is necessarily covered and hence Σ is well-covered.

The argument showing any collection of interval budgets is well-covered relied critically on the 'intermediate value' property of order intervals. We may relax this requirement by substituting a suitable comparability criterion between budgets. Recall that if $(X, \leq_X)$ is a lattice, a subset B dominates a subset B' in the strong set order if, for all $x \in B$ and $x' \in B'$, $x \vee x' \in B$ and $x \wedge x' \in B'$.

Example 5 (Comparability of Budgets)**.** Suppose $(X, \leq_X)$ is a lattice, and that Σ consists of totally ordered subsets. If every pair of budgets in Σ is comparable in the strong set order, then Σ is well-covered. For a formal proof, see Appendix I.

Finally, it is easy to construct new well-covered collections from existing ones. In particular, well-coveredness is preserved by the taking of certain restrictions.

Example 6 (Restrictions of Well-covered Environments)**.** Suppose (X, Σ) is well-covered, and $A \subseteq X$. Then $(X, \Sigma|_A)$ is well-covered, where $\Sigma|_A = \{B \in \Sigma : B \subseteq A\}$. This follows straightforwardly by observing that, if $\Sigma|_A$ were not well-covered, then its uncovered cyclic collections could not become covered by passing to Σ, as all the added budgets must contain alternatives that do not belong to A.

2.4 Relation to the Classical Demand Framework

The question of when the weak axiom of demand theory is empirically distinguishable from the strong axiom in the classical demand framework has a long history. Rose

(1958) first proved that these axioms coincide in the case of two goods, though Gale (1960) soon after established that this result did not hold for the case of three or more goods. In a recent contribution, Cherchye et al. (2018) characterized those linear budget collections for which several variants of the demand-theoretic weak and strong axioms coincide. Interestingly, they find that many widely used price-consumption datasets have large subsets exhibiting insufficient price variation to independently distinguish these axioms.[7] Given the apparent empirical shortcomings of field data for purposes of independently testing these phenomena, one is naturally led to consider how to construct simple, finite, laboratory experiments capable of rectifying this deficiency. Our Theorem 1 then provides a complete characterization of precisely which abstract choice experiments have testable implications of the generalized axiom in excess of the weak. Moreover, it is empirically and computationally desirable then to understand the problem for those environments in which one must take seriously indivisibilities, price non-linearities, or other economic phenomena contrary to the linear budget paradigm, which our model speaks to.

While Theorem 1 holds equally well when $X = \mathbb{R}^n_+$ and elements of Σ are linear budgets, our results neither imply nor are implied by those of Cherchye et al. (2018). We assume no intrinsic order structure on the set of alternatives, thus make no requirement of a rationalizing preference being monotone. Particularly, we allow for choice correspondences that do not satisfy Walras' law. As such, our paper holds for more general data sets where the chosen commodity bundle does not lie on the budget frontier, but as a consequence we require a purely choice-based notion of revealed

[7] They find that roughly 70% of the Spanish survey ECPF (Encuesta Continua de Presupestos Familiares) panel dataset (see, for example Beatty and Crawford (2011)) satisfies their condition for when a WARP-based analysis is equally informative as SARP-based. Even more drastically, roughly 97% of price triples in the British FES (Family Expenditure Survey) cross-sectional data set (see, for example, Blundell et al. (2003), Blundell et al. (2008), Blundell et al. (2015)) satisfy their condition for WARP and SARP to coincide.

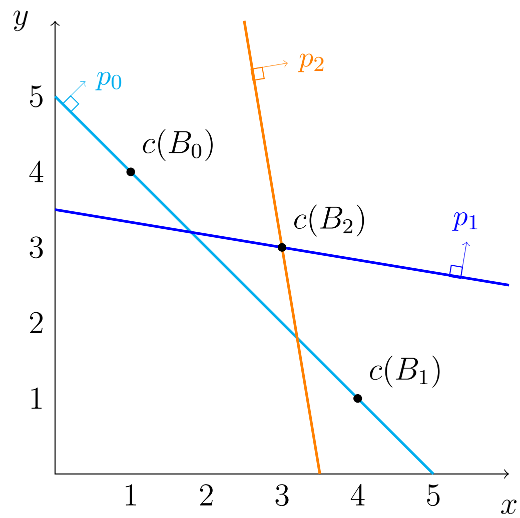


Figure 2: The frontiers of three linear budgets on $\mathbb{R}^2_+$. While the demand-theoretic weak and strong axioms coincide for any collection of linear budgets for two commodities, the above choices satisfy the choice-theoretic weak but not generalized axiom.

preference rather than one that makes use of the order structure of $\mathbb{R}^n$.[8] Additionally, we consider the solution concept of strong rationalizability, under which we require that the observed choices constitute the *entirety* of the agent's optimal choices from a given budget (classical references include Samuelson (1938), Houthakker (1950), Arrow (1959)). This leads to a different notion of which 'cycles' constitute violations of our notion of rationalizability, and hence to differing characterizations of which environments lead to such cycles inducing reversals, even when the class of budgets considered is the same (see Matzkin and Richter (1991), Nishimura et al. (2017)). In light of this, our results and those of Cherchye et al. (2018) are best thought of as complementary, addressing different frameworks and valid in differing contexts.

We conclude this section with an example of a collection of linear budgets in the

[8]A consequence of this, however, is that our theory requires, as part of our definition of an observation, a complete description of the budget from which an agent chose. Walras' Law, on the other hand, provides an identifying assumption to pin down the budget set for an observation from only price and consumption data.

two-commodity case (and hence for which the classical demand variants of the weak and strong axioms coincide) but which is not well-covered.

Example 7 (Non Well-covered Linear Budget Collection on $\mathbb{R}^2_+$)**.** Let $X = \mathbb{R}^2_+$, and consider three price tuples $p_0 = (\frac{1}{5}, \frac{1}{5})$, $p_1 = (\frac{1}{21}, \frac{1}{3.5}))$, and $p_2 = (\frac{1}{3.5}, \frac{1}{21})$. Let Σ consist of the three wealth-normalized linear budget sets formed by these price vectors: $B_i = B(p_i, 1)$ (see Fig. 2). Suppose an agent were to choose $c(B_0) = \{(1, 4)\}$, $c(B_1) = \{(4, 1)\}$, and $c(B_2) = \{(3, 3)\}$. These are all alternatives belonging to each budget and, with the exception of the choice from B_1 all lie on the budget frontier (recall that we do not impose Walras' law). Moreover, c satisfies the weak axiom but exhibits a three-cycle:

$$(3, 3) \succ_c (1, 4) \succ_c (4, 1) \succ_c (3, 3),$$

and hence the collection cannot be well-covered. This stands in comparison to Rose's result that the classical demand version of the weak axiom coincides with (weak) rationalizability in the two-commodity case, no matter the budget collection.

3 Abstract Choice and Integrability

3.1 Preliminaries

Well-coveredness of the budget collection is, in general, difficult to verify in practice, as it requires checking every cyclic collection for covering budgets. Without extra structure on the problem, this may become computationally difficult for larger experiments. Motivated by this difficulty, in this section we consider instead only those implications of well-coveredness that are reflected in the structure of the budget graph. If a budget collection is well-covered, clearly every loop in the budget graph of length four or more must possess a bisecting edge, that is an edge connecting vertices of the loop that does not belong to the loop's edge set. A graph with this property is said to be **chordal**. Critically, this property is efficiently verifiable: it is possible to determine whether a graph is chordal in linear time using standard methods (see, for example Rose et al. (1976)).

In this section, we consider experiments with only a chordal budget graph, a necessary, though not sufficient, condition for the well-coveredness of the collection. We show that an experiment possesses a chordal budget graph if and only if strong rationalizability coincides with (i) the weak axiom, and (ii) a mild, discrete analogue of the Slutsky symmetry axiom of differential demand theory. This serves as a trade-off relative to Theorem 1: in exchange for requiring somewhat more structure than just the weak axiom on the part of the choice data, one obtains an efficiently verifiable minimal richness condition on the choice environment for no 'small' cycles to imply no cycles of any kind.

Such results appear also in the mechanism design literature, where it is of great interest to have criteria on type spaces that guarantee testing the global condition of cyclic monotonicity (a cardinal form of the generalized axiom) reduces to testing only pairwise comparisons (e.g. Saks and Yu (2005), Ashlagi et al. (2010), analogous to our Theorem 1) or pairwise comparisons plus only 'local' no cycle conditions (e.g. Archer and Kleinberg (2014) and Kushnir and Lokutsievskiy (2019), which are analogous to our results in this section).

This result may be interpreted as an extension of integrability theory to the abstract choice framework. In particular, on rich enough domains our theory allows for incompletely observed, in particular potentially finite, data (corresponding to cases when Σ is far from complete) as opposed to the classical theory which takes as primitive a fully observed demand function. However, the relaxation to incomplete data can only go so far: our results also establish the chordality of the budget graph as the weakest possible richness condition on an environment under which strong rationalizability is equivalent to the classical integrability criteria of the weak axiom, plus a 'local no cycles' condition.

3.2 Abstract Analogues of the Integrability Conditions

Let (X, Σ) be a fixed choice environment, with $\Gamma(X, \Sigma)$ its budget graph. Let:

$$T_\Gamma = \big\{\{x, y, z\} \subseteq X : \{x, y\}, \{y, z\}, \{x, z\} \in E_\Gamma\big\}.$$

The combinatorial **domain** associated to the environment (X, Σ) is the triple $\mathcal{D}(X, \Sigma) = (X, E_\Gamma, T_\Gamma)$. The combinatorial domain essentially serves as a 'triangulation' of the set of alternatives using only the information encoded in the budget graph. For a choice correspondence $c \in \mathcal{C}(X, \Sigma)$ with revealed preference pair $(\succsim_c, \succ_c)$, we say that c is **locally rationalizable** if we may extend $(\succsim_c, \succ_c)$ by a single relation $\succeq$ such that:[9]

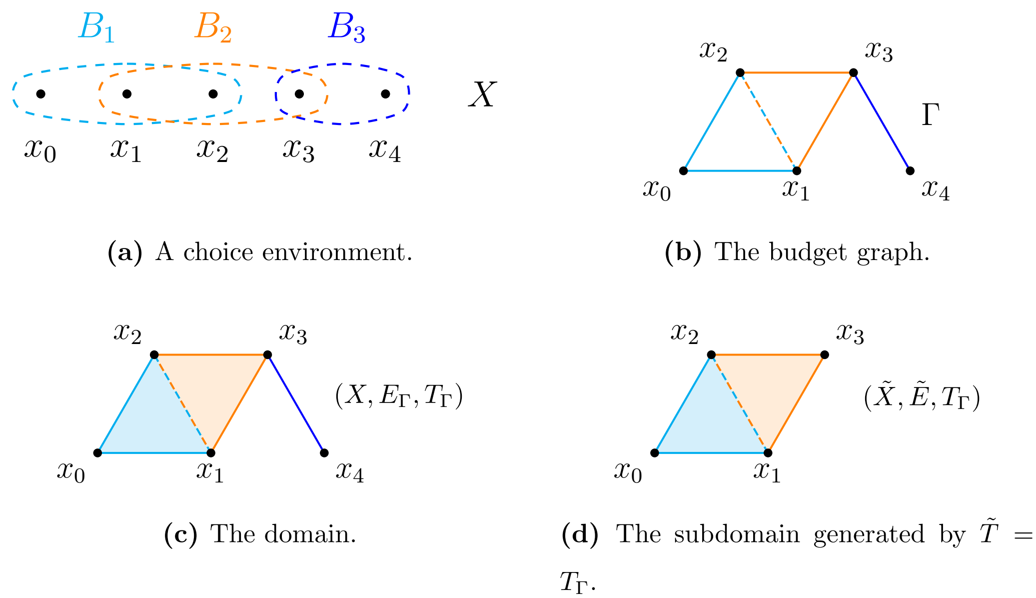
$$(\forall \tau \in T_\Gamma) \quad \succeq|_\tau \text{ is complete and transitive.}$$

Local rationalizability is the ordinal analogue of the joint conditions of Slutsky negative semi-definiteness and symmetry. It says nothing more than we may strongly rationalize the revealed preference locally, about each triangle in the domain, much the same way as the usual properties on the Slutsky matrix guarantee an economically suitable local solution to the system of differential equations defining the integrability problem.

Local rationalizability is necessary, though not sufficient, for the strong rationalizability of c (as any strongly rationalizing preference relation is a local rationalization). The fact that one must potentially consider an extension of $\succsim_c$ is simply a consequence of allowing for the possibility that Σ is highly incomplete and c does not reveal any preference between some pairs in some triangles of T_Γ.[10]

[9]Formally, we mean that $(\succeq, \succ)$ is an order pair extension of $(\succsim_c, \succ_c)$, where $\succ$ is the asymmetric component of $\succeq$.

[10]If, for example, $E_\Gamma \subseteq \Sigma$, there would be no need to consider an extension. In particular, on a complete domain, no extension is ever necessary.



(a) A choice environment.

(b) The budget graph.

(c) The domain.

(d) The subdomain generated by $\tilde{T} = T_\Gamma$.

Figure 3: The various constructions associated with a choice environment. The shading of the triangles is intended to indicate their inclusion in T_Γ.

3.2.1 A Decomposition of Local Rationalizability

We may decompose the property of local ratinonalizability into two properties: the weak axiom, and a property we term ordinal irrotationality, which serves as the analogue of Slutsky symmetry for the abstract choice model. For a choice correspondence $c \in \mathcal{C}(X, \Sigma)$ with revealed preference pair $(\succsim_c, \succ_c)$, we say that c is **ordinally irrotational** if the revealed preference pair $(\succsim_c, \succ_c)$ admits an order pair extension $(\succeq, \succ^*)$ of its revealed preference such that:

$$(\forall \tau \in T_\Gamma) \quad (\succeq, \succ^*)|_\tau \text{ is complete and has no three-cycle,}$$

and for which $\succ^*$ is asymmetric and contains the asymmetric component of $\succeq$. Much the same as Slutsky symmetry, ordinal irrotationality is a 'no local cycles' condition, plus some minor regularity. Moreover, in conjunction with the weak axiom, it characterizes local rationalizability.

Proposition 1. *Let $c \in \mathcal{C}(X, \Sigma)$. Then c is locally rationalizable if and only if c both:*

(i) obeys the weak axiom; and

(ii) is ordinally irrotational.

The weak axiom and ordinal irrotationally are also logically independent, as the next example shows.

Example 8 (Independence of WARP and OI)**.** Let $X = \{x_0, x_1, x_2, x_3\}$, and suppose Σ consists of two budgets, $B_1 = \{x_0, x_1, x_2\}$, and $B_2 = \{x_1, x_2, x_3\}$. Let $c(B_1) = \{x_1, x_2\}$ and $c(B_2) = \{x_2\}$. Then c does not satisfy the weak axiom: $x_1 \succsim_c x_2$ but $x_2 \succ_c x_1$. However, c is ordinally irrotational: let $\succeq = \succsim_c \cup \{(x_1, x_3)\}$ and $\succ^* = \succ_c \cup \{(x_1, x_3)\}$. Then $\succ \subsetneq \succ^*$, $\succ^*$ remains asymmetric, and the restriction of $(\succeq, \succ^*)$ to each triangle in the domain (here, given by the two budgets) contains no three-cycle.[11]

3.3 Sampling & Integrability

The satisfaction of the generalized axiom of revealed preference by a choice correspondence implies its strong rationalizability and hence both the weak axiom and ordinal irrotationality, no matter the structure of the domain. However, the sufficiency of the weak axiom and ordinal irrotationality for the generalized axiom depends crucially on the structure of the domain. Intuitively, the denser the budget graph (that is, the greater the number of pairs of alternatives the experiment is capable of revealing a preference between), then the more triangles there will be and hence the more stringent the requirement of local rationalizability will be.

This is not the whole story, however. What turns out to be most important is the manner in which the triangles of the budget graph fit together. Certain collections of triangles may fit together in ways that permit even cyclic revealed preferences to be

[11]This establishes that OI does not imply WARP. To rule out the converse implication, it suffices to consider a three-element set of alternatives with three binary budget sets and any cyclic revealed preference.

locally rationalizable (e.g. Figure 4). What is needed is that there be enough 'good' collections of triangles in the budget graph to cover every loop, allowing the condition of local rationalizability to rule out any possible cycles. This turns out to be possible, precisely when the budget graph of the environment is chordal. For any environment with a chordal budget graph, if a choice correspondence (i) obeys the weak axiom, and (ii) is ordinally irrotational, then it is also strongly rationalizable. Moreover, possessing a chordal budget graph, it turns out, is the weakest possible richness condition on Σ under which *any* such traditional integrability result can possibly hold: for any environment without a chordal budget graph there always exist choice correspondences which obey the weak axiom and are ordinally irrotational (and hence locally rationalizable), yet nonetheless are not strongly rationalizable.

Theorem 2. *Let* (X, Σ) *be a choice environment with* $\Gamma(X, \Sigma)$ *chordal. Then a choice correspondence* $c \in \mathcal{C}(X, \Sigma)$ *is strongly rationalizable if and only if:*

(i) It obeys the weak axiom; and

(ii) It is ordinally irrotational.

Moreover, (i) and (ii) are jointly equivalent to the strong rationalizability of c *if and only if* $\Gamma(X, \Sigma)$ *is chordal.*

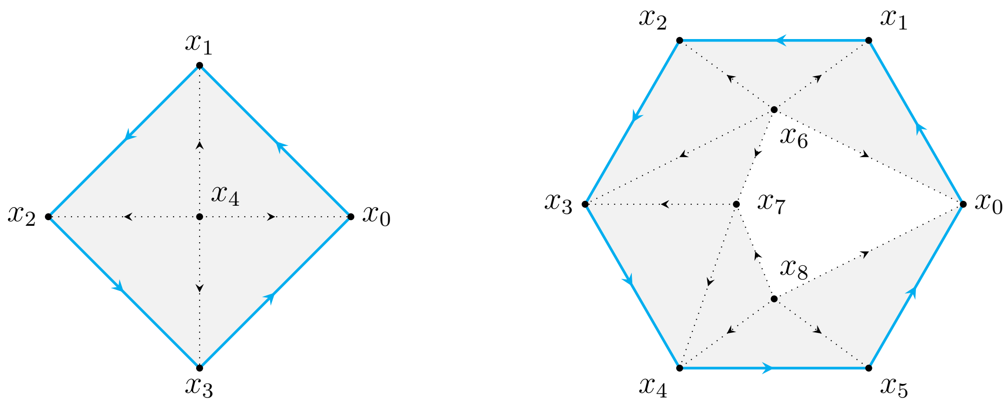
As local rationalizability is always necessary for strong rationalizability, regardless of the structure of the domain, we obtain the following corollary.

Corollary 1. *Let* (X, Σ) *be a choice environment, and suppose* $\Gamma(X, \Sigma)$ *is not chordal. Then there exists a choice correspondence which is locally, but not strongly, rationalizable.*

We emphasize that $\Gamma(X, \Sigma)$ being chordal is a requirement of sufficient ex-ante richness of the choice environment. A given pair of alternatives is connected by an edge in the budget graph if and only if there exists a choice correspondence on that environment capable of revealing a preference between them. Chordality specifically requires that, given any GARP violation in the data, that the experiment is rich enough that it is capable of revealing a preference between some non-adjacent pair of alternatives belonging to the cycle.[12]

In comparison with the classical integrability theory, we critically do not suppose a complete domain, or even an infinite data set. We allow for arbitrary environments, and our results fully characterize just what is needed observationally for the usual integrability conditions of the weak axiom and ordinal irrotationality to extend. In particular, Theorem 2 imposes no analytic, point-set, or order-theoretic assumptions on model primitives. This is particularly notable given the historical program of attempting to weaken the differentiability hypotheses of the classical integrability theory, e.g. Berger and Meyers (1966), Hartman (1970), and Berger and Myers (1971). Nonetheless, there is a cost to this generality. In the classical theory, while one supposes a great deal more structure on the budgeter or demand function and its domain, one obtains a rationalizing utility with commensurately fine properties. Comparatively, all that is guaranteed by Theorem 2 is a rationalizing weak order. This is the price, it appears, of an integrability theorem that not only allows for finite data sets, but also imposes no hypotheses that are non-falsifiable by such data sets. Put another way, Theorem 2 operates wholly within the empirical content of abstract choice model, in the sense of Chambers et al. (2014).

[12]This contrasts with well-coveredness, which corresponds to the condition that, for any GARP violation, the experiment is rich enough to *guarantee* either a WARP violation, or a that some preference is revealed between a non-adjacent pair in the cycle (see Lemmas 2 and 3).



(a) A cyclic revealed preference (blue) with a local rationalization (black) on a topologically trivial subdomain that is not combinatorially trivial.

(b) A cyclic revealed preference (blue) and local rationalization (black) on a combinatorially trivial subdomain. The domain however retracts onto its outer boundary loop, hence it is topologically non-trivial.

Figure 4: When subdomains fail to be either combinatorially trivial or topologically trivial, the criterion of local rationalizability is too weak to guarantee the absence of cycles.

3.4 Proof Sketch

3.4.1 Simple Domains

While conceptually simple to state, the property of an environment having a chordal budget graph is a difficult global property to work with for purposes of proving Theorem 2. We first establish an equivalent characterization of chordality, in terms of collections of triangles in the associated domain, that is more suited to our purposes. Let $\tilde{T} \subseteq T_\Gamma$ be a collection of triangles in the budget graph. Let $\tilde{X}$ denote the points of X contained within some element of $\tilde{T}$, and $\tilde{E}$ the subset of edges in E_Γ contained in some element of $\tilde{T}$. Then the **subdomain** generated by $\tilde{T}$ is defined as the tuple:

$$\mathcal{D}(X, \Sigma)|_{\tilde{T}} = (\tilde{X}, \tilde{E}, \tilde{T}).$$

We will be particularly interested in subdomains possessing specific structure. For a finite $\tilde{T}$, we say the subdomain generated by $\tilde{T}$ is **combinatorially trivial** if, for every pair $\tau, \tau' \in \tilde{T}$, there is a unique sequence of of distinct elements of $\tilde{T}$:

$$\tau = \tau_1, \tau_2, \ldots, \tau_k = \tau'$$

where, for each $1 \leq j \leq k-1$, the triangles τ_j and τ_{j+1} share precisely a pair of common elements. If one imagines an undirected graph whose nodes are the elements of $\tilde{T}$ and whose edge relation is given by sharing a pair of common elements, then combinatorial triviality amounts to asking the graph associated with the collection $\tilde{T}$ be a tree.

Similarly, for a finite collection $\tilde{T}$ we say that the subdomain generated by $\tilde{T}$ is **topologically trivial** if it has no 'holes' in it in an appropriate sense.[13] If one imagines a subdomain as consisting of a graph whose triangles are 'filled in' forming a kind of triangulated surface, topological triviality roughly asks that this surface be simply connected or contractible.

We will say that a subdomain is **simple** if it is both combinatorially and topologically trivial. A loop $\gamma = (V_\gamma, E_\gamma)$ in the budget graph is contained in a subdomain $(\tilde{X}, \tilde{E}, \tilde{T})$ if $V_\gamma \subseteq \tilde{X}$ and $E_\gamma \subseteq \tilde{E}$. By abuse of notation, we will say the entire domain is simple if every loop in it is contained in a simple subdomain. This turns out to be equivalent to requiring that every loop in the budget graph possessing a bisecting edge.

Theorem 3. *Let (X, Σ) be a choice environment. Then the domain $\mathcal{D}(X, \Sigma)$ is simple if and only if the budget graph is chordal.*

3.4.2 Discrete Calculus

We now argue that, on a simple domain, every choice correspondence satisfying (i) the weak axiom, and (ii) ordinal irrotationality is strongly rationalizable. To do this, we will recast our ordinal problem into a cardinal form that shares strong formal

[13]Formally, we say a subdomain $\mathcal{D}|_{\tilde{T}}$ is topologically trivial if it has a first Betti number of zero, i.e. $H_1(\mathcal{D}|_{\tilde{T}}; \mathbb{R}) = 0$, where H_* denotes simplicial homology with real coefficients. See Munkres (1984) p. 34.

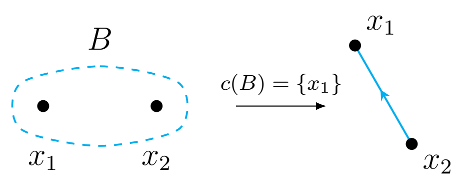


Figure 5: Given a choice correspondence, we may view the relations of its revealed preference as specifying ordinal 'flows' along edges of the budget graph. The budget graph is the smallest network such that this interpretation remains valid for any choice correspondence.

similarities with the classical differential approach. In particular, we make use of the discrete exterior calculus, which is most suitable for our combinatorial structure.

A vector field, or 1-form, on the domain $\mathcal{D} = (X, E_\Gamma, T_\Gamma)$ is a map $F : \hat{E}_\Gamma \to \mathbb{R}$ such that $F(x, y) = -F(y, x)$, where $\hat{E}_\Gamma = \{(x, y) \in X \times X : \{x, y\} \in E_\Gamma\}$.[14] We interpret such a map as describing a magnitude of flow from x to y, and where a negative flow is simply interpreted as a flow in the opposite direction. Similarly, a 0-form on $\mathcal{D}$ is simply an element of $\mathbb{R}^X$, and a 2-form a map $\mathfrak{F} : \hat{T}_\Gamma \to \mathbb{R}$ such that $\mathfrak{F}(x_{\sigma(0)}, x_{\sigma(1)}, x_{\sigma(2)}) = \text{sign}(\sigma) \cdot \mathfrak{F}(x_0, x_1, x_2)$ for any permutation σ.

There are natural operators between the spaces of forms on $\mathcal{D}$. For any 0-form f, we define the gradient of f to be the 1-form defined by:

$$\text{grad}(f)(x, y) = f(y) - f(x).$$

Similarly, for a 1-form F, its rotation, or curl, is defined pointwise as the 2-form:

$$\text{rot}(F)(x, y, z) = F(x, y) + F(y, z) + F(z, x).$$

It is straightforward to verify that both the gradient and curl are linear operators, and that the image of the gradient operator is vector subspace of the kernel of the curl operator. In particular, we term a 1-form F integrable if it belongs to the image of the gradient operator and irrotational if it belongs to the kernel of the rotation. One may

[14]We let $\hat{E}_\Gamma$ (resp. $\hat{T}_\Gamma$) denote the spaces of *oriented* edges (resp. triangles) of the budget graph.

think of the curl operator as measuring how far from (cardinally) transitive a given 1-form is about each triangle in the budget graph.

We will be interested in the existence of an integrable vector field that is consistent with the revealed preference pair (e.g. Figure 5). Let $(\succsim_c, \succ_c)$ denote the revealed preference of a choice correspondence c. We say that a 1-form F is a **cardinalization** of c if:

$$y \succsim_c x \implies F(x, y) \geq 0,$$

and

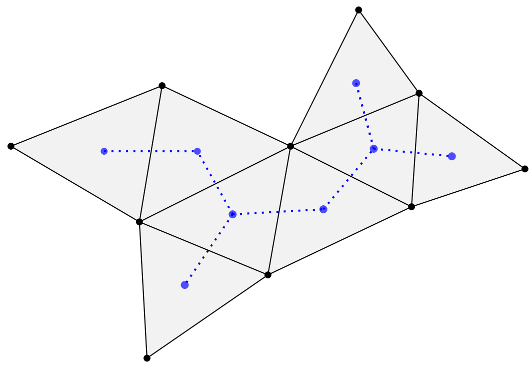
$$y \succ_c x \implies F(x, y) > 0.$$

For any choice correspondence c, let K_c denote the set of all 1-forms on $\mathcal{D}$ cardinalizing $(\succsim_c, \succ_c)$. For any c, K_c is a convex cone. Moreover, K_c is non-empty if and only if c obeys the weak axiom.[15]

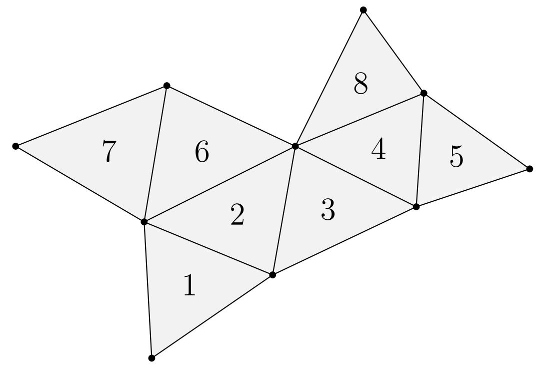
3.4.3 Proof Sketch

Consider a subdomain $(\tilde{X}, \tilde{E}, \tilde{T})$ generated by some arbitrary finite collection $\tilde{T}$ of triangles, and consider some (for sake of exposition) asymmetric, locally rational $\succ_c$. To define a cardinalization F on this domain, it suffices to choose the values of F on those ordered pairs (x, y) where $x \succ_c y$, as this uniquely determines the value of F on all ordered pairs of the form (y, x). By this implicit choice of basis, we are identifying the cone K_c with the interior of the positive orthant of the space of 1-forms. We first consider the problem of whether there exists an irrotational cardinalization, a necessary though generally not sufficient condition for the existence of an integrable

[15] Cardinalizations as we have defiend them are intimately related to the idea of preference functions. See, for example Shafer (1974), Quah (2006), or Aguiar et al. (2020). Unlike previous work, however, we do not assume that the cardinalization is defined for all pairs in X, but rather only those in $\hat{E}_\Gamma \subseteq X \times X$.



(a) A collection of triangles that is combinatorially trivial. The collection is combinatorially trivial becauses the 'sharing an edge' graph (dotted blue) is a tree.



(b) A good ordering of the triangles, guaranteeing an irrotational cardinalization for any locally rational binary relation.

Figure 6: On combinatorially trivial subdomains, one may always enumerate the triangles such that any triangle shares at most a single edge with the union of those triangles preceding it. The existence of such an enumeration guarantees a simple algorithm can construct irrotational cardinalizations for any locally rational binary relation.

cardinalization:[16]

$$\operatorname{rot} F = 0$$

$$F \gg 0.$$

Combinatorial triviality ensures a solution to this problem for any locally rational binary relation. Consider the subdomain of Fig. 6. Call an enumeration of the triangles in the subdomain a 'good ordering' if it has the property that for any triangle in the enumeration, its intersection with the collection of all preceding triangles consists of at most a single edge. The combinatorial triviality of a subdomain guarantees good orderings exist. Such orderings provide a roadmap to constructing an irrotational cardinalization: restricting any one triangle, an irrotational cardinalization exists, simply by local rationality of $\succ_c$. Consider now any adjacent triangle. Again by local ratio-

[16]We employ the vector notation $x \geq y$ to denote that x is component-wise larger than y. We will write $x > y$ to denote that x is component-wise greater with at least one component strictly so, and $x \gg y$ to denote that every component of x strictly exceeds that of y.

nality we can find an irrotational cardinalization for just this triangle. However, we can form an irrotational cardinalization on the subdomain generated by both triangles together by ensuring the values of the flows on each triangle agree on the common edge the triangles share. This can always be attained by choosing irrotational but otherwise arbitrary flows on each triangle then multiplying the flows on one of the two by an appropriate positive scalar. More generally, in any good ordering no triangle shares more than a single edge with the collection of all preceding triangles, and thus an inductive application of this argument guarantees an irrotational cardinalization for any locally rational relation on a combinatorially trivial subdomain.[17]

Thus we are guaranteed that for any locally rational $\succ_c$, on any combinatorially trivial subdomain, the cone of consistent cardinalizations K_c intersects the subspace of irrotational cardinalizations $\ker(\mathrm{rot})$. Indeed as K_c is the interior of the positive orthant of the space of flows, the intersection $\tilde{K}_c = K_c \cap \ker(\mathrm{rot})$ will be of full dimension in $\ker(\mathrm{rot})$, and we may choose a basis for this subspace identifying $\tilde{K}_c$ with the interior of its positive orthant. Since the image of the gradient is a subspace of the kernel of the curl, we may view the gradient as a linear map taking a 0 form to an *irrotational* 1-form. Thus, given our choice of basis for $\ker(\mathrm{rot})$, the existence of an integrable cardinalization is equivalent existence of a $u \in \mathbb{R}^{\tilde{X}}$ such that:

$$\mathrm{grad}\, u \gg 0.$$

By Gordan's Alternative (see Gordan (1873)), exactly one of the following holds: (i)

[17]Without combinatorial triviality, irrotational cardinalizations are not guaranteeed exist, even for locally rational binary relations; see, for example, Figure 4.(a). The locally rational relation on Figure 4.(b) does admit an irrotational cardinalization (the subdomain is combinatorially trivial), albeit not an integrable one.

the above problem admits a solution, or (ii) there exists a 1-form F such that:

$$\operatorname{grad}^\top F = 0$$
$$F > 0$$
$$\operatorname{rot} F = 0.$$

In particular, if no integrable cardinalization of $\succ_c$ exists, there is some non-zero $F \in \ker(\operatorname{rot}) \cap \ker(\operatorname{grad}^\top)$. However, this possibility is precisely ruled out by topological triviality, which ensures that this intersection is $\{0\}$. For any two matrices $A \in \mathbb{R}^{l\times m}$ and $B \in \mathbb{R}^{m\times n}$ such that $AB = 0$, there exist isomorphisms:

$$\ker(A) \cap \ker(B^\top) \cong \ker(A^\top A + BB^\top) \cong \ker(B^\top)/\operatorname{im}(A^\top).$$

Thus, in particular:

$$\ker(\operatorname{rot}) \cap \ker(\operatorname{grad}^\top) \cong \ker(\operatorname{grad}^\top)/\operatorname{im}(\operatorname{rot}^\top).$$

The reader familiar with simplicial homology will recognize that $\operatorname{grad}^\top$ is ∂_1 and $\operatorname{rot}^\top$ is ∂_2, that is they are the homological boundary operators between the appropriate spaces of real-valued chains on the simplicial complex $(\tilde{X}, \tilde{E}, \tilde{T})$. In particular, the homology group (here, vector space) of the subdomain in dimension 1 with real coefficients is precisely $\ker(\operatorname{grad}^\top)/\operatorname{im}(\operatorname{rot}^\top)$, and hence topological triviality implies $\ker(\operatorname{rot}) \cap \ker(\operatorname{grad}^\top) = \{0\}$. Thus for any locally rational relation on a combinatorially and topologically trivial subdomain, there exists an integrable cardinalization, precluding the existence of cycles supported on any loop contained in the subdomain. On a simple domain, every loop is contained in some such subdomain, allowing us to guarantee the generalized axiom holds.

3.5 Examples of Environments With Chordal Budget Graphs

Possessing a chordal budget graph is a much weaker notion than nearly any existing 'completeness' criterion for choice problems. Indeed, most notion of completeness actually yield a budget graph that is the complete graph on X, a significantly stronger condition.

Example 9 (Complete Abstract Environments)**.** Recall that for a general choice environment (X, Σ), the budget collection Σ is complete if it contains all finite subsets of X. If Σ is complete, then clearly its budget graph is complete as well.

Another example of a class of normatively complete environments that have a complete budget graph (and hence a simple domain) are complete collections of linear budgets.

Example 10 (Complete Collections of Linear Budgets)**.** Suppose $X = \mathbb{R}^L_+$ and Σ consists of all (income normalized) linear budgets $B(p, 1) = \{x \in \mathbb{R}^+_L : \langle p, x\rangle \leq 1\}$ for all $p \in \mathbb{R}^L_{++}$. In light of the argument in the complete abstract environments example, it suffices to show that every pair of distinct vectors of commodities forms an edge in the budget graph. Consider $x, y \in \mathbb{R}^L_+$, $x \neq y$ and let $x \vee y$ denote their component-wise supremum. Let $B(\tilde{p}, 1)$ denote any linear budget containing $x \vee y$; then $B(\tilde{p}, 1)$ contains both x and y and hence $\{x, y\} \in E_\Gamma$. Thus the budget graph is again complete, and hence chordal.

More generally, the linearity of the budgets played no role verifying the simplicity of the domain, yielding a natural generalization to the broader class of budget sets considered by Forges and Minelli (2009).

Example 11 (Complete Forges-Minelli Environments)**.** Let $X = \mathbb{R}^L_+$. We will term Σ a complete Forges-Minelli budget collection if (i) every $B \in \Sigma$ is compact and there exists some increasing, continuous $g_B : \mathbb{R}^L_+ \to \mathbb{R}$ such that $B = \{x \in \mathbb{R}^L_+ : g_B(x) \leq 0\}$, and (ii) for every $x \in \mathbb{R}^L_+$, there exists some $B \in \Sigma$ such that $x \in B$. The budget graph associated with any complete Forges-Minelli environment is complete: for any distinct $x, y \in \mathbb{R}^L_+$, by (ii) there exists some $\tilde{B} \in \Sigma$ containing $x \vee y$. By (i), $\tilde{B}$ is downward closed, hence $x, y \in \tilde{B}$ too.

Our next example is of collections of budgets that satisfy a natural notion of completeness but nonetheless yield a budget graph that is generally less-than-complete.

Example 12 (Identifying Experiments)**.** Let X be a locally compact Polish space. We consider those budget collections studied by Chambers et al. (2020) in the context of the non-parametric identification of continuous preferences. These consist of those collections Σ that consist of (i) a countable collection of binary sets, (ii) such that $\cup_{B\in\Sigma}B$ is dense in X, and (ii) such that for all $x, y \in \cup_{B\in\Sigma}B$, $\{x, y\} \in \Sigma$. When X is uncountable, the budget graph will contain uncountably many isolated vertices. However, for any such experiment by (i) and (iii) the budget graph will be the complete graph on some countable subset of vertices (along with its isolated vertices) which nonetheless is chordal.

Of course, Theorems 1 and 2 are intimately related and, in particular, any well-covered environment has a simple domain.

Example 13 (Well-covered Abstract Environments)**.** If (X, Σ) is a general choice environment with Σ well-covered, then the budget graph is chordal

4 Applications

4.1 Cardinality-Constrained Choice

The complete cardinality-constrained problem, in essence considered by Arrow (1959), is one of the most well-known examples of a domain on which the weak axiom is equivalent to strong rationalizability: if Σ contains every subset of X of cardinality at most three, then the weak axiom suffices for the generalized. While Theorem 1 of provides necessary and sufficient conditions with or without a cardinality constraint, for this special case Theorem 2 also provides an intuitive means of characterizing specifically which collections of small budgets are well-covered.

Suppose Σ contains no budget of cardinality greater than three, and denote the sub-collection of three-element budgets by $\Sigma_3 \subseteq \Sigma$. Our first observation is that, for the cardinality-constrained case, the weak axiom implies local rationalizability if and only if every triangle in the budget graph is itself a budget.

Proposition 2. *Let (X, Σ) be a cardinality-constrained choice environment. Then every choice correspondence c that obeys the weak axiom is locally rationalizable if and only if $T_\Gamma = \Sigma_3$.*

Notably, this holds independently of the structure of the budget graph. Making use of this, Theorem 2 immediately provides the following characterization of when the weak and generalized axioms coincide for domains consisting of small budgets.

Corollary 2. *Let (X, Σ) be a cardinality-constrained choice environment. Then the weak axiom characterizes strong rationalizability for any choice correspondence if and only (i) $T_\Gamma = \Sigma_3$, and (ii) the budget graph $\Gamma(X, \Sigma)$ is chordal.*

4.2 Deterministic Rationalizability of Stochastic Choice

There has been recent interest in deterministic notions of rationality that could be ascribed to models of stochastic choice. Let X be a finite set, and Σ a not-necessarily-complete collection of budgets. In this context, we take as primitive a collection of probability distributions $\mathbb{P}(\cdot, B)$, for each $B \in \Sigma$, corresponding to the observed frequency with which a given alternative is chosen when an agent is presented with choice set B.

Ok and Tserenjigmid (2019) put forward two choice correspondences arising from such stochastic data. The 'upper' choice correspondence associated with $\mathbb{P}$ maps a budget to those alternatives in it that are observed to be chosen with positive probability:

$$C^{\mathbb{P}}(B) = \{x \in B : \mathbb{P}(x, B) > 0\}.$$

The 'lower' correspondence returns only those alternatives that are chosen with maximal frequency:

$$C_{\mathbb{P}}(B) = \{x \in B : \forall y \in B,\ \mathbb{P}(x,B) \geq \mathbb{P}(y,B)\}.$$

The authors term $\mathbb{P}$ completely upper (resp. lower) rational if $C^{\mathbb{P}}$ (resp. $C_{\mathbb{P}}$) is strongly rationalizable by a preference relation and obsesrve that, when Σ is complete, these rationality properties are characterized by $C^{\mathbb{P}}$ and $C_{\mathbb{P}}$ satisfying the weak axiom. Theorem 1 provides an immediate extension of these results to any well-covered budget collection, allowing considerable latitude by extending the results to experiments with less-than-complete domains. This is particularly valuable in the stochastic context, as to obtain reasonable empirical estimates of $\mathbb{P}$, one must sample each budget in Σ repeatedly. As such, reductions in the required breadth of Σ may lead to considerable savings in terms of the observational requirements of the theory.

4.3 Aggregation of Incomplete Preferences

Consider a set of national policies X, and let $\mathcal{I}$ be a set of agents. Given a subset of $A \subseteq X$, let $\mathcal{P}(A)$ denote the set of all preference relations on A. For each agent i, a regional preference is a relation $\succsim_i \in \mathcal{P}(S_i)$ for some fixed $S_i \subseteq X$. We interpret this as capturing that agents care only about policies affecting their particular region, or perhaps those neighboring regions. We term the tuple $(X, (S_i)_{i\in\mathcal{I}})$ a society. A social welfare function, for a given society, is simply a map $F : \prod_i \mathcal{P}(S_i) \to \mathcal{P}(X)$.

We say a social welfare function F satisfies the **Pareto** axiom if, whenever all agents who have preferences between two policies x and y agree on their relative ranking, this ranking is preserved by F. Notably, unlike in the case of complete preferences, the incompleteness of preferences may well lead to the set of Pareto social welfare functions being empty. When agents have preferences over only subset(s) of policies, it may be the case that every aggregation mechanism is forced to disregard the preferences of some populations, even when their views are unopposed. Moreover, this may be true even for profiles of (incomplete) preferences containing *no disagagreement of any kind.*

A social welfare function is said to satisfy the **strong unanimity** axiom if, whenever there is no disagreement over any pair of policies by any pair of agents, the social welfare function respects the preferences of the agents. Formally, F satisfies strong unanimity if, whenever for all $i, j \in \mathcal{I}$ it is the case that $\succsim_i |_{S_i \cap S_j} = \succsim_j |_{S_i \cap S_j}$, then $F(\succsim_1, \ldots, \succsim_N)$ is an order extension of $\cup_i \succsim_i$. Strong unanimity is far weaker condition on F than satisfying the Pareto axiom; indeed it requires the Pareto axiom to hold only for special case of unanimous profiles. Nonetheless, the regional nature of the preferences may lead to the set of social welfare functions that satisfy even just strong unanimity being empty. Theorem 2 provides a complete characterization of those societies for which the set of social welfare functions that satisfy strong unanimity is non-empty.

Let $\mathcal{S}$ be a society. In this setting, define the domain associated with $\mathcal{S}$ as the triple $\mathcal{D}(\mathcal{S}) = (X, E_S, T_S)$, where E_S (resp. T_S) are those pairs (resp. triples) of distinct policies belonging to some common S_i. We interpret this as follows: to each agent i, suppose we solicit their pairwise preferences on each binary choice set within S_i. Since we may restrict to profiles that are unanimous, there is agreement over $S_i \cap S_j$ for all pairs of agents, and since each agent i is rational over S_i, the union of the revealed preference relations is precisely a locally rational binary relation on our domain. Thus Theorem 2 implies that if (X, E_S) is chordal, there is always some weak order extending the union of these preferences, and we may always define the value of a social welfare function F at such a profile to be one of these extensions. Conversely, if (X, E_S) is not chordal, Theorem 2 guarantees a profile of unanimous regional preferences admitting no extending weak order, precluding the existence of any F satisfying the strong unanimity axiom for such a society.

Corollary 3. *Let $\mathcal{S} = (X, (S_i)_{i \in \mathcal{I}})$ be a society. The set of social welfare functions for $\mathcal{S}$ satisfying the strong unanimity axiom is non-empty if and only if $\mathcal{D}(\mathcal{S})$ is simple.*

Proofs

Lemma. *Let (X, Σ) be a choice environment and let γ be a loop in $\Gamma(X, \Sigma)$. Then there exists choice function $c \in \mathcal{W}(X, \Sigma)$ such that $\succsim_c |_{E_\gamma}$ is a cycle if and only if there exists a cyclic collection $\mathcal{B}_\gamma$ and choice function $\tilde{c} \in \mathcal{W}(X, \Sigma|_{\mathcal{B}_\gamma})$ such that $\succsim_{\tilde{c}} |_{E_\gamma}$ is a cycle.*

Proof. ($\Longrightarrow$): Suppose there exists a $c \in \mathcal{W}(X, \Sigma)$ such that $\succsim_c |_{E_\gamma}$ is a cycle. Then there exists some cyclic collection $\mathcal{B}_\gamma$ with the property that the choices inducing $\succsim_c |_{E_\gamma}$ are all made on elements of $\mathcal{B}_\gamma$. Then the restriction of c to $\Sigma|_{\mathcal{B}_\gamma}$ must still obey the weak axiom, and clearly satisfies the conclusion of the lemma.

($\Longleftarrow$): Suppose now there exists a cyclic collection $\mathcal{B}_\gamma$ and a $\tilde{c} \in \mathcal{W}(X, \Sigma|_{\mathcal{B}_\gamma})$ such that $\succsim_{\tilde{c}} |_{E_\gamma}$ is a cycle. Define an extension of $\tilde{c}$ to all of Σ as follows:

$$c(B) = \begin{cases} \tilde{c}(B) & \text{if } B \in \Sigma|_{\mathcal{B}_\gamma} \\ B \setminus \left(\cup_{\tilde{B} \in \mathcal{B}_\gamma} \tilde{B} \right) & \text{else.} \end{cases}$$

This defines a choice correspondence in $\mathcal{W}(X, \Sigma)$, for if $x \succsim_c y$ for distinct x, y, either $x, y \in \cup_{\tilde{B} \in \mathcal{B}_\gamma} \tilde{B}$, in which case there can be no violation of the weak axiom as $\tilde{c}$ is in $\mathcal{W}\big(X, \Sigma|_{\mathcal{B}_\gamma}\big)$, or $x \notin \cup_{\tilde{B} \in \mathcal{B}_\gamma} \tilde{B}$, in which case by construction $\neg\, y \succ_c x$, and thus $c \in \mathcal{W}(X, \Sigma)$. $\square$

Lemma. *Let (X, Σ) be a choice environment and let γ be a loop in $\Gamma(X, \Sigma)$ with $|V_\gamma| = 3$. Then there exists a choice correspondence $c \in \mathcal{W}(X, \Sigma)$ with $\succsim_c |_{E_\gamma}$ a cycle if and only if there exists a cyclic collection $\mathcal{B}_\gamma$ that is not covered.*

Proof. ($\Longleftarrow$): Suppose that $\mathcal{B}_\gamma$ is an uncovered cyclic collection for γ of minimal cardinality. Let us denote $E_\gamma = \{e_0, e_1, e_2\}$. Then, in particular, for every $e_j \in E_\gamma$, there is a unique $B_j \in \mathcal{B}_\gamma$ with $e_j \subseteq B_j$. Define $\tilde{c} \in \mathcal{C}(X, \Sigma|_{\mathcal{B}_\gamma})$ via:

$$\tilde{c}(B) = \begin{cases} e_j \cap e_{j+1} & \text{if } \exists\, e_j \in E_\gamma \text{ s.t. } B \cap V_\gamma = e_j \\ B \cap V_\gamma & \text{if } |B \cap V_\gamma| = 1 \\ B & \text{else.} \end{cases}$$

where all subscripts are taken mod-3. Note $\tilde{c}$ is well-defined, as $\mathcal{B}_\gamma$ is uncovered from which it follows the first two cases exhaust the possibilities for budgets in $\Sigma|_{\mathcal{B}_\gamma}$ that intersect V_γ. Moreover, $\tilde{c} \in \mathcal{W}(X, \Sigma|_{\mathcal{B}_\gamma})$. First, observe the restriction of the pair $(\succsim_{\tilde{c}}, \succ_{\tilde{c}})|_{E_\gamma}$ satisfies the weak axiom. But the only alternatives $\tilde{c}$ reveals strictly preferred to any others all lie in V_γ, and the only goods ever revealed preferred to elements of V_γ also lie in V_γ. Hence $\tilde{c} \in \mathcal{W}\big(X, B \in \Sigma|_{\mathcal{B}_\gamma}\big)$, and by Lemma 1 there exists a $c \in \mathcal{W}(X, \Sigma)$ such that $\succsim_c |_{E_\gamma}$ is cyclic.

($\Longrightarrow$): Let $c \in \mathcal{W}(X, \Sigma)$ be such that $\succsim_c |_{E_\gamma}$ is cyclic. Then there exists a cyclic collection $\mathcal{B}_\gamma$ on which choices generating the cycle $\succsim_c |_{E_\gamma}$ are made; fix such a collection. We now show that this cyclic collection must be uncovered, lest there exist some $B \in \Sigma|_{\mathcal{B}_\gamma}$ such that $V_\gamma \subseteq B$. Suppose, for sake of contradiction, that such a B exists.

Case 1: Suppose first that $c(B) \cap V_\gamma \neq \varnothing$. Then either $c(B)$ induces complete indifference across V_γ, or there exists some pair of elements of V_γ that is either strictly preferred to, or strictly dominated by the third element. Both possibilities preclude the existence of the cycle $\succsim_c |_{E_\gamma}$ for any $c \in \mathcal{W}(X, \Sigma)$.

Case 2: Suppose then that $c(B) \cap V_\gamma = \varnothing$: then for all $x \in V_\gamma$ and $y \in c(B)$ we have $y \succ_c x$. But $c(B) \subset B \subseteq \cup_{\tilde{B} \in \mathcal{B}_\gamma} \tilde{B}$, and since for all $x \in V_\gamma$ there exists some $\tilde{B}$ such that $x \in c(\tilde{B})$, there exists an $\tilde{x} \in V_\gamma$ and $\tilde{B} \in \mathcal{B}_\gamma$ such that $\tilde{x}, y \in \tilde{B}$ and $\tilde{x} \in c(\tilde{B})$. This contradicts our hypothesis that $c \in \mathcal{W}(X, \Sigma)$. $\square$

Lemma. *Let (X, Σ) be a choice environment and let γ be a loop in $\Gamma(X, \Sigma)$ with $|V_\gamma| > 3$. Suppose there exists a choice correspondence $c \in \mathcal{W}(X, \Sigma)$ with $\succsim_c |_{E_\gamma}$ a cycle. If every cyclic collection $\mathcal{B}_\gamma$ is covered, then there exists a loop γ' in $\Gamma(X, \Sigma)$*

such that $|V'_\gamma| < |V_\gamma|$ *and with* $\succsim_c |_{E_{\gamma'}}$ *a cycle.*

Proof. Let $\mathcal{B}_\gamma$ be a minimal cyclic collection on which choices inducing $\succsim_c |_{E_\gamma}$ are made, and suppose $\mathcal{B}_\gamma$ is covered. Then there exists some $B \in \Sigma|_{\mathcal{B}_\gamma}$ such that B contains a non-adjacent pair of vertices of γ. We proceed in two cases.

Case 1: Suppose first that $c(B)$ does not intersect V_γ. Let $x_k, x_{k'} \in B \cap V_\gamma$ be one such non-adjacent pair of vertices, and let $y \in c(B)$. As $c(B) \subseteq B \subseteq \cup_{\tilde{B} \in \mathcal{B}_\gamma} \tilde{B}$, and $\mathcal{B}_\gamma$ is a minimal cyclic collection on which choices inducing the cycle $\succsim_c |_{E_\gamma}$ are made, there is some $\tilde{B}_{k^*} \in \mathcal{B}_\gamma$ containing y, such that there is some $x_{k^*} \in c(\tilde{B}_{k^*}) \cap V_\gamma$. Without loss of generality, let $x_{k'} \succsim_c \cdots \succsim_c x_{k^*} \succsim_c \cdots \succsim_c x_k$. In particular, by our hypothesis that c obeys the weak axiom, we cannot have $x_{k^*} = x_k$ (or $x_{k'}$).[18] As $c(B)$ does not contain any element of V_γ by hypothesis, but $x_{k'} \in B$, we have $y \succ_c x_{k'}$, and, as $x_{k^*}, y \in \tilde{B}_{k^*}$, it follows $x_{k^*} \succsim_c y$. Thus: $y \succ_c x_{k'} \succsim_c \cdots x_{k^*} \succsim_c y$. Define γ' to be the graph with $V_{\gamma'}$ given by the above collection of points, and $E_{\gamma'}$ consisting of those pairs related in the above cycle (clearly as there is a non-empty revealed preference for each pair this forms a loop in $\Gamma(X, \Sigma)$). By construction, $\succsim_c |_{E_{\gamma'}}$ is a cycle. Now, since $x_{k^*} \neq x_k$, $x_k \notin V_{\gamma'}$. Moreover, since x_k and $x_{k'}$ are non-adjacent in γ, under $\succsim_c |_{E_\gamma}$ we also have: $x_k \succsim_c \cdots \succsim_c \bar{x} \succsim_c \cdots \succsim_c x_{k'}$ along the 'other side' of the loop. Thus we also have that $\bar{x} \notin V_{\gamma'}$. So while we have added a point y not in V_γ to our $V_{\gamma'}$, we have omitted at least two others, x_k and $\bar{x}$, and we conclude: $|V_{\gamma'}| < |V_\gamma|$ as required.

Case 2: Suppose now that $c(B)$ intersects V_γ. As B contains the non-adjacent pair $x_k, x_{k'} \in V_\gamma$, the only way that $c(B)$ can avoid revealing a preference between x_k and $x_{k'}$ is if neither is in but both are adjacent in γ to $c(B)$. Moreover, this argument holds for every non-adjacent pair of vertices of γ contained in B. Now, if $c(B)$ induces a revealed preference $x_i \succsim_c x_j$ between any pair of non-adjacent vertices $x_i, x_j \in V_\gamma$ this partitions $\succsim_c |_{E_\gamma}$ into two sub-cycles, one of which must always contain a strict relation (either from $\succsim_c |_{E_\gamma}$ or resulting from a strict revealed preference between x_i and x_j).

[18] As $y \succ_c x_k$ and $y \succ_c x_{k'}$ by hypothesis, but $x_{k^*} \succsim_c y$ via choice on B_{k^*}.

Letting γ' be defined by the vertices and pairs supporting any such sub-cycle suffices to prove the claim. Thus suppose that $c(B)$ does not induce any revealed preference between any non-adjacent pair (lest we be done). Thus $c(B)$ is adjacent to both x_k and $x_{k'}$ (and hence singleton) and $c(B) = \{x^*\}$ induces both $x_k \prec_c x^* \succ_c x_{k'}$. But these three points are all elements of V_γ, hence by virtue of $\succsim_c |_{E_\gamma}$ being a cycle we have either $x_k \succsim_c x^* \succsim_c x_{x'}$ or the reverse. But both of these yield contradiction via a violation of the weak axiom, and hence there exists a strictly shorter $\succsim_c$-cycle. □

Theorem. *Let (X, Σ) be a choice environment. Then $\mathcal{W}(X, \Sigma) = \mathcal{G}(X, \Sigma)$ if and only if Σ is well-covered.*

Proof. ($\Longleftarrow$): For purposes of contraposition, suppose that $\mathcal{W}(X, \Sigma) \neq \mathcal{G}(X, \Sigma)$. Then there exists some loop γ in the budget graph $\Gamma(X, \Sigma)$ and some choice correspondence $c \in \mathcal{W}(X, \Sigma)$ such that $\succsim_c |_{E_\gamma}$ is a cycle. If $|V_\gamma| = 3$, then by Lemma 2, Σ is not well-covered and we are done. Hence suppose γ is of length strictly greater than three. Then there exists some cyclic collection $\mathcal{B}_\gamma$ on which choices generating the cycle $\succsim_c |_{E_\gamma}$ are made. If $\mathcal{B}_\gamma$ is not covered, we are done, hence suppose it is. Then by Lemma 3 there exists a loop γ' in the budget graph of strictly shorter length such that $\succsim_c |_{E_{\gamma'}}$ is also a cycle. As we have already concluded this process cannot repeat until it hits a three-cycle, we conclude that at some stage, there exists some loop $\gamma^{(n)}$ for which there exists a cyclic collection $\mathcal{B}_{\gamma^{(n)}}$ which is not covered and hence Σ is not well-covered.

($\Longrightarrow$): We again proceed by contraposition. If a cyclic collection for a budget graph loop of length 3 is uncovered, by Lemma 2, we immediately obtain $\mathcal{W}(X, \Sigma) \neq \mathcal{G}(X, \Sigma)$. Suppose then there exists some loop γ with $|V_\gamma| > 3$ with a cyclic collection $\mathcal{B}_\gamma$ that is uncovered (without loss of generality, let $\mathcal{B}_\gamma$ be a minimal such uncovered cyclic collection) In particular, let $E_\gamma = \{e_0, \ldots, e_{J-1}\}$. By virtue of γ being uncovered, for each $e_j \in E_\gamma$ there exists a $\tilde{B}_j \in \mathcal{B}_\gamma$ such that for all $j \in \{0, \ldots, J-1\}$ we have $e_j = \tilde{B}_j \cap V_\gamma$, and by the minimality of $\mathcal{B}_\gamma$, these $\{\tilde{B}_j\}$ are unique and completely exhaust $\mathcal{B}_\gamma$. Furthermore, for all $B \in \Sigma|_{\mathcal{B}_\gamma}$, $B \cap V_\gamma$ necessarily also either equals some

e_j, is singleton, or is empty.[19] Thus, letting (subscripts taken mod-J):

$$\tilde{c}(B) = \begin{cases} e_j \cap e_{j+1} & \text{if } \exists\, e_j \text{ s.t. } e_j = B \cap V_\gamma \\ B \cap V_\gamma & \text{if } |B \cap V_\gamma| = 1 \\ B & \text{else,} \end{cases}$$

we obtain a choice correspondence $c \in \mathcal{W}(X, \Sigma|_{\mathcal{B}_\gamma})$ by an argument identical to that in the proof of Lemma 2, only for a longer cycle. Clearly $\succsim_{\tilde{c}} |_{E_\gamma}$ is cyclic and by Lemma 1 this extends to a choice correspondence in $c \in \mathcal{W}(X, \Sigma)$ such that $\succsim_c |_{E_\gamma}$ is cyclic, and hence $\mathcal{W}(X, \Sigma) \neq \mathcal{G}(X, \Sigma)$. Thus, by contraposition, $\mathcal{W}(X, \Sigma) = \mathcal{G}(X, \Sigma)$ implies the well-coveredness of Σ. □

Corollary. *Let $(X, \leq_X)$ be a lattice, and suppose further that (i) Σ contains only totally ordered subsets of X, and (ii) every pair of elements of Σ is comparable in the strong set order. Then Σ is well-covered.*

Proof. Let γ denote an arbitrary loop in $\Gamma(X, \Sigma)$. By (i) we conclude that every edge pair of γ is related by $\leq_X$. As $\leq_X$ is a partial order and V_γ finite, $\leq_X |_{E_\gamma}$ admits a local minimum, in the sense of the existence of $x_{i-1}, x_i, x_{i+1} \in V_\gamma$ such that $x_i <_X x_{i-1}, x_{i+1}$, and $\{x_{i-1}, x_i\}, \{x_i, x_{i+1}\} \in E_\gamma$. Let $\mathcal{B}_\gamma$ be an arbitrary cyclic collection for γ. In light of (ii), without loss of generality suppose these two edges belong to different budgets in $\mathcal{B}_\gamma$, and let $B_{x_{i-1}x_i} \leq_{SSO} B_{x_i x_{i+1}}$ for two budgets in $\mathcal{B}_\gamma$ with $B_{x_{i-1}x_i}$ containing $\{x_{i-1}, x_i\}$ and $B_{x_i x_{i+1}}$ containing $\{x_i, x_{i+1}\}$. Then by the strong set order $x_{i-1} = x_i \vee x_{i-1} \in B_{x_{i+1}x_i}$, and hence $B_{x_{i+1}x_i}$ covers $\mathcal{B}_\gamma$. As γ and $\mathcal{B}_\gamma$ were arbitrary, we again conclude that Σ is well-covered. □

Combinatorial Results on Simple Subdomains

We begin by recalling some definitions from the theory of simplicial complexes. A **simplicial complex** is a set of vertices $\{v_i\}_{i \in \mathcal{I}}$, and collection of non-empty finite

[19] The loop γ, viewed as a loop in the subgraph $\Gamma(X, \Sigma|_{\mathcal{B}_\gamma})$, is what is sometimes referred to as 'chordless' in graph theory.

subsets $\{s_j\}_{j\in\mathcal{J}}$ of $\{v_i\}$ called **simplices** such that:

1. Any set consisting of exactly one vertex is a simplex; and

2. Any non-empty subset of a simplex is a simplex.[20]

A simplex of cardinality $(n+1)$ is said to be of dimension n. Given a simplicial complex $\mathcal{D}_T$ generated by a collection of triangles T, the **boundary** $\mathcal{D}_T$, denoted $\dot{\mathcal{D}}_T$, is the subcomplex of generated by those 1-simplices which are the faces of exactly one triangle of T. The n-**skeleton** of a complex K, denoted $K^{(n)}$ is defined as the collection of all simplices of K of dimension n. We will commit the mild sin of occasionally using $K^{(n)}$ to denote both the set of n-simplices of K and also the subcomplex generated by these simplices where no confusion should result.

Theorem (Fundamental Theorem of Simple Sub-domains). *Let $\mathcal{D}$ be an arbitrary domain, and $l \subseteq \mathcal{D}^{(1)}$ a loop. There exists a simple sub-domain for l if and only if there exists a simple sub-domain $\mathcal{D}|_{\bar{T}}$ for l that satisfies:*

(i) **Boundary***: $\dot{\mathcal{D}}|_{\bar{T}} = l$; and*

(ii) **Minimality***: The vertex set of $\mathcal{D}|_{\bar{T}}$ equals that of l.*

Proof. $(\Longleftarrow)$: Trivial.

$(\Longrightarrow)$: Let $\tilde{T}$ generate a simple sub-domain for l, and consider a chain $\lambda \in C_1(\mathcal{D}|_{\tilde{T}}, \mathbb{R})$, given by:

$$\lambda = \sum_{\sigma\in l} n_\sigma \sigma,$$

with (i) zero coefficients on any $\sigma \in \mathcal{D}|_{\tilde{T}}$ that does not belong to l, and (ii) and such that, for all $\sigma \in l$, the coefficients satisfy $|n_\sigma| = 1$, where signs are chosen so $\lambda \in \ker \partial_1$.[21] As, by topological triviality, $H_1(\mathcal{D}|_{\tilde{T}}, \mathbb{R}) = 0$, $\operatorname{Im}\partial_2 = \ker\partial_1$ hence there

[20] See Spanier (1989) Section 3.1 (p. 108) for basic definitions.

[21] The apparent indeterminacy of the signs of the coefficients in λ is simply a consequence of our being ambivalent about the choice basis for $C_1(\mathcal{D}|_{\tilde{T}}, \mathbb{R})$.

exists some chain Λ in $C_2(\mathcal{D}|_{\tilde{T}}, \mathbb{R})$ solving:

$$\partial_2 \underbrace{\left[\sum_{\tau \in \tilde{T}} n_\tau \tau \right]}_{\Lambda} = \lambda$$

with some n_τ possibly equal to zero. Let $\bar{T} = \{\tau \in \tilde{T} : |n_\tau| \neq 0\}$ denote the support of Λ, and suppose for some 2-simplex $\tau \in \mathcal{D}|_{\bar{T}}$ there is a 1-face $\hat{\sigma}$ of τ such that $\hat{\sigma} \in \dot{\mathcal{D}}|_{\bar{T}}$ but $\hat{\sigma} \notin l$. Then we immediately obtain a contradiction: it must be the case actually $n_\tau = 0$, since $\partial_2 \Lambda$ would have coefficient equal in absolute value to $|n_\tau|$ on $\hat{\sigma}$, as by definition of a boundary, τ is the only 2-simplex in $\mathcal{D}|_{\bar{T}}$ containing $\hat{\sigma}$, and $n_{\hat{\sigma}} = 0$ as $\hat{\sigma} \notin l$. Therefore we conclude Λ is supported on a finite sub-collection $\bar{T}$ with the property that $\dot{\mathcal{D}}|_{\bar{T}} \subseteq l \subseteq \mathcal{D}|_{\bar{T}}$.

We claim first that $\mathcal{D}|_{\bar{T}}$ is combinatorially trivial. Suppose for sake of contradiction, that it fails to be so. Since $\tilde{T}$ was combinatorially trivial and $\bar{T} \subseteq \tilde{T}$, there exist a partition of $\bar{T}$ into maximal, non-empty collections of 2-faces $\bar{T}_1, \ldots, \bar{T}_K$, $K > 1$, such that for each k, $\mathcal{D}|_{\bar{T}_k}$ is combinatorially trivial. This in turn implies that for all k, $\dot{\mathcal{D}}|_{\bar{T}_k} \neq \varnothing$, as any 'leaf' 2-face contains at least two 1-faces unique to it. Fix an arbitrary k and let $\hat{\Lambda}$ be a 2-chain in $C_2(\mathcal{D}|_{\bar{T}_k}, \mathbb{R})$ whose coefficients are are all 1 in absolute value, with signs chosen so that $\partial_2 \hat{\Lambda}$ vanishes on any 1-face not in $\dot{\mathcal{D}}|_{\bar{T}_k}$. Then by construction,

$$\partial_2 \hat{\Lambda} = \sum_{\sigma \in \dot{\mathcal{D}}|_{\bar{T}_k}} \hat{n}_\sigma \sigma$$

and for all such σ, $|\hat{n}_\sigma| = 1$. By identity, $(\partial_1 \circ \partial_2)(\hat{\Lambda}) = 0$, and thus for each vertex $x \in \dot{\mathcal{D}}|^{(0)}_{\bar{T}_k} \neq \varnothing$, x is contained in an even number of 1-faces in $\dot{\mathcal{D}}|_{\bar{T}_k}$, and hence $\dot{\mathcal{D}}|_{\bar{T}_k}$ consists of a union of loops. But as $\dot{\mathcal{D}}|_{\bar{T}_{\hat{k}}} \subsetneq \dot{\mathcal{D}}|_{\bar{T}} \subseteq l$, we obtain a contradiction, as no proper subcomplex of a loop may be a loop. Thus $\mathcal{D}|_{\bar{T}}$ is itself combinatorially trivial.

We now verify that $\dot{\mathcal{D}}|_{\bar{T}} = l$. Recall we have already obtained that $\dot{\mathcal{D}}|_{\bar{T}} \subseteq l \subseteq \mathcal{D}|_{\bar{T}}$. Suppose then, for sake of contradiction, that there exists some 1-face $\sigma \in l$, such that

$\sigma \notin \dot{\mathcal{D}}|_{\bar{T}}$. Let $\tau \in \mathcal{D}|_{\bar{T}}$ denote one of the two 2-faces (combinatorial triviality) of $\mathcal{D}|_{\bar{T}}$ that contains σ, and let K denote the sub-complex of $\mathcal{D}|_{\bar{T}}$ generated by those 2-faces of $\mathcal{D}|_{\bar{T}}$ that may be reached from τ by a sequence of distinct 2-simplices with adjacent terms sharing a common face, but whose intersections do not contain σ. By construction K is combinatorially trivial; by an argument analogous to that of the preceding paragraph, $\dot{K}$ is a non-empty union of loops. But $\dot{K} \subsetneq \dot{\mathcal{D}}|_{\bar{T}} \cup \{\sigma\} \subseteq l$, where the first strict inclusion follows from the fact that the complement of K in $\mathcal{D}|_{\bar{T}}$ is a non-empty combinatorially trivial subcomplex too. Hence we obtain a contradiction, again because l cannot contain any proper sub-complex that is also a loop, and thus $\dot{\mathcal{D}}|_{\bar{T}} = l$ as claimed.

We turn to verifying our minimality claim, that the vertex sets of $\mathcal{D}|_{\bar{T}}$ and l coincide: $\mathcal{D}|_{\bar{T}}^{(0)} = l^{(0)}$. Let G denote the undirected graph whose vertex set is given by the 2-faces of $\mathcal{D}|_{\bar{T}}$ and whose edge set determined by the relation of having an intersection containing a 1-face. By combinatorial triviality of $\mathcal{D}|_{\bar{T}}$, G is a tree. Now, suppose toward a contradiction that the vertex sets of $\mathcal{D}|_{\bar{T}}$ and l do not coincide. Since $\dot{\mathcal{D}}|_{\bar{T}} = l$, this implies there is some vertex x of $\mathcal{D}|_{\bar{T}}$ not in l. Now, as $\mathcal{D}|_{\bar{T}}$ is combinatorially trivial, every 1-face σ of $\mathcal{D}|_{\bar{T}}$ that contains x is contained in precisely two 2-simplices. Let $\tilde{G}$ be the subgraph of G consisting of those 2-faces containing x as a vertex. Since each vertex τ of $\tilde{G}$ contains precisely two 1-faces that contain x, by finiteness $\tilde{G}$ is a cycle graph, contradicting the fact that G is a tree (i.e. that $\mathcal{D}|_{\bar{T}}$ is combinatorially trivial). Hence the vertex sets of $\mathcal{D}|_{\bar{T}}$ and l coincide.

Finally, we show the dimension-1 simplicial homology of $\mathcal{D}|_{\bar{T}}$ is zero in real coefficients, our last outstanding claim. As $\mathcal{D}|_{\bar{T}}$ is combinatorially trivial, its collection of 1-faces may be partitioned into two subsets: those faces in $\dot{\mathcal{D}}|_{\bar{T}}$ and those not. By definition, the edge-set of the graph G introduced in the preceding paragraph is in one-to-one correspondence with the the set of 1-faces of $\mathcal{D}|_{\bar{T}}$ not in $\dot{\mathcal{D}}|_{\bar{T}}$. By combinatorial triviality, G is a tree and hence has one more vertex (2-simplex of $\mathcal{D}|_{\bar{T}}$) than edge (1-face of $\mathcal{D}|_{\bar{T}}$ not in $\dot{\mathcal{D}}|_{\bar{T}}$). Similarly, $l = \dot{\mathcal{D}}|_{\bar{T}}$ is a loop, so the number of 1-faces must be the same as the number of vertices of l, which we have established is also the vertex set of $\mathcal{D}|_{\bar{T}}$. The Euler-Poincaré theorem (Munkres (1984) Theorem 22.2) asserts the equivalence of the following two definitions of the Euler characteristic of $\mathcal{D}|_{\bar{T}}$:

$$\chi(\mathcal{D}|_{\bar{T}}) = \dim H_0(\mathcal{D}|_{\bar{T}}, \mathbb{R}) - \dim H_1(\mathcal{D}|_{\bar{T}}, \mathbb{R}) + \dim H_2(\mathcal{D}|_{\bar{T}}, \mathbb{R}) = V - E + F,$$

where V is the number of 0-simplices, E the number of 1-simplices, and F the number of 2-simplices in $\mathcal{D}|_{\bar{T}}$. By the above counting argument for the set of 1-faces of $\mathcal{D}|_{\bar{T}}$, we know:

$$E = \underbrace{V}_{\text{1-faces in } \dot{\mathcal{D}}|_{\bar{T}}.} + \underbrace{F - 1}_{\text{1-faces in } \mathcal{D}|_{\bar{T}} \setminus \dot{\mathcal{D}}|_{\bar{T}}.} \tag{1}$$

and hence $\chi(\mathcal{D}|_{\bar{T}}) = 1$. Now, since every 2-simplex in $\mathcal{D}|_{\bar{T}}$ intersects l, $\mathcal{D}|_{\bar{T}}$ is path-connected and hence $\dim H_0(\mathcal{D}|_{\bar{T}}, \mathbb{R}) = 1$. Moreover, by combinatorial triviality, $H_2(\mathcal{D}|_{\bar{T}}, \mathbb{R}) = 0$.[22] Then by Euler-Poincaré, $\dim H_1(\mathcal{D}|_{\bar{T}}, \mathbb{R}) = 0$, and thus $\mathcal{D}|_{\bar{T}}$ is a simple sub-domain as claimed. □

Lemma (Union Lemma)**.** *Let $\mathcal{D}|_T, \mathcal{D}|_{T'}$ be two simple sub-domains whose intersection consists of a single 1-face σ. Then $\mathcal{D}|_T \cup \mathcal{D}|_{T'}$ is a simple sub-domain.*

Proof. As $\mathcal{D}|_T$ and $\mathcal{D}|_{T'}$ are combinatorially trivial, it is immediate that so too is $\mathcal{D}|_T \cup \mathcal{D}|_{T'}$. Then, by the reduced simplicial Mayer-Vietoris theorem (Munkres (1984)

[22] Since $\mathcal{D}|_{\bar{T}}$ is homengeously 2-dimensional it contains no simplices of dimension greater than two, hence $H_2(\mathcal{D}|_{\bar{T}}, \mathbb{R}) = 0$ if and only if the only solution to:

$$\partial_2 \Bigg[\sum_{\tau \in \mathcal{D}|_{\bar{T}}^{(2)}} \tilde{n}_\tau \tau \Bigg] = 0$$

is for $\tilde{n}_\tau = 0$ for all $\tau \in \mathcal{D}|_{\bar{T}}^{(2)}$. Clearly for any solution, any $\tau \in \mathcal{D}|_{\bar{T}}$ which contains a 1-face in $\dot{\mathcal{D}}|_{\bar{T}}$ must have have $\tilde{n}_\tau = 0$ by (PM.2). Hence in any non-zero solution to the above, the sub-collection of 2-simplices in $\mathcal{D}|_{\bar{T}}$ with non-zero coefficients must have the property that all of their 1-faces are contained also in some other (hence unique other) member of the sub-collection. But this sub-collection defines a subgraph of the graph G, and the above property implies again that this subgraph can have no leaves, contradicting the fact G is a tree, as $\mathcal{D}|_{\bar{T}}$ is combinatorially trivial. Hence $H_2(\mathcal{D}|_{\bar{T}}, \mathbb{R}) = 0$.

Theorem 25.1) there exists an exact sequence:

$$0 \to \tilde{H}_1(\mathcal{D}|_T \cap \mathcal{D}|_{T'}, \mathbb{R}) \to \tilde{H}_1(\mathcal{D}|_T, \mathbb{R}) \oplus \tilde{H}_1(\mathcal{D}|_{T'}, \mathbb{R}) \to \tilde{H}_1(\mathcal{D}|_T \cup \mathcal{D}|_{T'}, \mathbb{R}) \to \tilde{H}_0(\mathcal{D}|_T \cap \mathcal{D}|_{T'}, \mathbb{R})$$

which, making use of topological triviality of $\mathcal{D}|_T$ and $\mathcal{D}|_{T'}$ and the contractibility of $\mathcal{D}|_T \cap \mathcal{D}|_{T'}$ (i.e. $= \sigma$), reduces to:

$$0 \to (0) \to (0) \oplus (0) \to \tilde{H}_1(\mathcal{D}|_T \cup \mathcal{D}|_{T'}, \mathbb{R}) \to 0$$

and hence $\tilde{H}_1(\mathcal{D}|_T \cup \mathcal{D}|_{T'}, \mathbb{R}) = 0$, and equivalently $H_1(\mathcal{D}|_T \cup \mathcal{D}|_{T'}, \mathbb{R})$ by definition of reduced simplicial homology. $\square$

We now turn to the proof of Theorem 3 in the text.

Theorem. *Let (X, Σ) be a choice environment. Then the domain $\mathcal{D}(X, \Sigma)$ is simple if and only if the budget graph is chordal.*

Proof. ($\Longrightarrow$): Suppose the domain is simple, and let γ be a loop. By simplicity of the domain, there exists some simple subdomain containing γ. By the Fundamental Theorem of Simple Subdomains, there exists a collection of triangles $\tilde{T}$ such that the edge-set of the subdomain generated by $\tilde{T}$ consists either of edges of γ or bisections of γ. If $|E_\gamma| = 3$ then there is nothing to check thus suppose $|E_\gamma| > 3$. Then by combinatorial triviality, there exists at least one element of $\tilde{E}$ that does not belong to E_γ, and thus γ has a chord. Since γ was arbitrary, we conclude $\Gamma(X, \Sigma)$ is chordal.

($\Longleftarrow$): To prove the domain associated to any chordal budget graph is simple, we proceed by contraposition. Suppose, then, that $\mathcal{D}(X, \Sigma)$ is not simple. Then, as there exists a loop contained in no simple sub-domain, there exists a shortest such loop, which we will denote γ, with $E_\gamma = \big\{\{x_0, x_1\}, \ldots, \{x_{n-1}, x_n\}, \{x_n, x_0\}\big\}$. We know $|V_\gamma|$ must be strictly greater than three, lest V_γ be a triple in $\mathcal{D}$ and hence this triple serve trivially as a simple sub-domain containing γ. We now prove that for all $e \in E_\Gamma$, $e \subseteq V_\gamma$ if and only if $e \in E_\gamma$, that is, that γ is a chordless loop in $\Gamma(X, \Sigma)$. Clearly $E_\gamma \subseteq E_\Gamma$. Thus, for sake of contradiction, suppose there exists an $e \in E_\Gamma$ with $e \subseteq V_\gamma$

but $e \notin E_\gamma$. Then without loss of generality, $e = \{x_j, x_k\}$ with $k > j + 1$. Hence we obtain two loops, γ_1 and γ_2 via:

$$E_{\gamma_1} = \big\{\{x_0, x_1\}, \ldots, \{x_{j-1}, x_j\}, \{x_j, x_k\}, \{x_k, x_{k+1}\}, \ldots, \{x_n, x_0\}\big\}$$

and

$$E_{\gamma_2} = \big\{\{x_j, x_{j+1}\}, \ldots, \{x_{k-1}, x_k\}, \{x_k, x_j\}\big\},$$

both shorter than γ. By the minimality of γ, there exist simple sub-domains $\mathcal{D}|_{\tilde{T}_1}$ and $\mathcal{D}|_{\tilde{T}_2}$ of $\mathcal{D}(X, \Sigma)$ for γ_1 and γ_2 respectively, and by the fundamental theorem of simple sub-domains, these complexes may be taken to intersect only on the 1-face $\{x_j, x_k\}$. But by the union lemma, $\mathcal{D}|_{\tilde{T}_1} \cup \mathcal{D}|_{\tilde{T}_2}$ is a simple sub-domain for γ, a contradiction. Thus γ is chordless, and hence Γ is not chordal. $\square$

Finally, we conclude with the (counter-)example mentioned in the text.

Example 14. Let $X = \{x_0, \ldots, x_4\}$, and let $\Sigma = E_\Gamma = \big\{\{x_0, x_1\}, \{x_1, x_2\}, \{x_2, x_3\}, \{x_3, x_4\}, \{x_4, x_0\}, \{x_1, x_3\}, \{x_3, x_0\}, \{x_0, x_2\}, \{x_2, x_4\}, \{x_4, x_1\}\big\}$. Note the domain associated to this environment corresponds to a triangulation of the Möbius strip. Let γ correspond to the boundary of the strip, i.e. the loop with edge set $E_\gamma = \big\{\{x_1, x_3\}, \{x_3, x_0\}, \{x_0, x_2\}, \{x_2, x_4\}, \{x_4, x_1$ As $X = V_\gamma$, clearly every edge in E_Γ consists of either an edge of E_γ or a bisecting edge, and it is simple to verify that every vertex belongs to some bisecting edge of γ. Finally, there is only one subdomain containing γ, the entire domain itself, and this is neither combinatorially trivial (its 'sharing a common face graph is a circle graph on five vertices) nor topologically trivial (the homology group of the Möbius strip in dimension one with $\mathbb{R}$-coefficients is $\mathbb{R}$).

Integrability Results

Given a simplicial complex K, a (discrete) n**-form** is a linear functional acting on oriented pieces of the n-dimensional skeleton $K^{(n)}$.[23] Define the space of all n-forms

[23] We intentionally adopt the analyst's terminology of 'forms' rather than the topologist's 'cochain' to further highlight the parallel to the exterior calculus arguments underpinning the solution to the

on K as:

$$C^n(K) = \{\phi : K^{(n)} \to \mathbb{R} : \phi([x_{\sigma(0)}, \ldots, x_{\sigma(n)}]) = \mathrm{sign}(\sigma)\phi([x_0, \ldots, x_n])\},$$

where $[x_0, \ldots, x_n]$ denotes an oriented n-simplex of K and σ is any permutation of $\{0, \ldots, n\}$. In particular, the vector space $C^0(K)$ consists precisely of all real valued functions on the vertices; $C^1(K)$ may be interpreted as the space of all real-valued flows on the 1-skeleton of K, where the permutation condition simply ensures that these flows are directed .

The gradient and curl operators are defined analogously as in the text. A 1-form F is said to be **exact** (or 'integrable') if there exists an $f \in C^0(K)$ such that $\mathrm{grad}(f) = F$. Similarly, if $\mathrm{rot}(F) = 0$, F is said to be **closed**. An exact 1-form is always closed; this may be succinctly stated as $\mathrm{Im}(\mathrm{grad}) \subseteq \mathrm{Ker}(\mathrm{rot})$. In particular, this implies the quotient vector space $\mathrm{Ker}(\mathrm{rot})/\mathrm{Im}(\mathrm{grad})$ is well-defined. This quotient is denoted $H^1(K, \mathbb{R})$ and is known as the first simplicial cohomology group of K (with $\mathbb{R}$-coefficients); its dimension may be interpreted as a measure of how far the closedness of a 1-form is from guaranteeing its exactness, or integrability.

Proposition. *Let $c \in \mathcal{C}(X, \Sigma)$. Then c is locally rationalizable if and only if c both:*

(i) obeys the weak axiom; and

(ii) is ordinally irrotational.

Proof. ($\Longrightarrow$): Suppose c is locally rationalizable. Then $\succeq$ and its asymmetric component $\succ$ form a suitable order pair extension to verify ordinal irrotationality. Moreover, c must obey the weak axiom for all those pairs of alternatives that are contained in some triangle of the budget graph. Thus we need only to verify that c does not violate the weak axiom for those pairs of distinct alternatives x, y which form edges not belonging to any triangle. But if $\{x, y\} \in E_\Gamma$ and $\{x, y\}$ is not contained in any triangle,

classical integrability problem. See Jiang et al. (2011) and Grady and Polimeni (2010) for an in-depth discussion of the parallels between the smooth and discrete theories.

then $\{x, y\} \in \Sigma$ and this must be the only budget containing both these alternatives, precluding any possible violation of the weak axiom over them.

($\Longleftarrow$): Suppose now c is ordinally irrotational and satisfies the weak axiom. Let $(\succeq, \succ^*)$ denote the order extension of the revealed preference guaranteed by ordinal irrotationality. In particular, the asymmetric component $\succ$ of $\succeq$ is a sub-relation of $\succ^*$, i.e. $\succ \subseteq \succ^*$. If $\succ = \succ^*$, then $\succsim$ is a local rationalization, thus it suffices to establish that one may always take $\succ^*$ to be the asymmetric component of $\succeq$.

Let $Z = \{(x, y) \in X \times X : x \succ^* y, y \succeq x, \text{ and } x \succeq y\}$ denote those pairs for which $x \succ^* y$ but $\neg x \succ y$ (recall that while $\succ^*$ contains $\succ$, by the definition of an order pair, $\succ^* \subseteq \succeq$). We partition Z into two subsets: $Z_0 = \{(x, y) \in Z : y \succsim_c x\}$ and $Z_1 = \{(x, y) \in Z : \neg y \succsim_c x\}$. Consider first those pairs in Z_0. If $(x, y) \in Z_0$ then $x \succ^* y$, $x \succeq y$, $y \succeq x$, and $y \succsim_c x$. As c satisfies the weak axiom, it cannot be the case that $x \succ_c y$. Moreover, it cannot be the case that $y \succ_c x$, as $\succ^*$ contains $\succ_c$ and $x \succ^* y$ already, and $\succ^*$ is asymmetric by hypothesis. Thus it must be that $y \succsim_c x$ and $x \succsim_c y$. In other words, both (x, y) and (y, x) are required to belong to $\succeq$, but we may omit (x, y) from $\succ^*$ without problem: $(\succeq, \succ^* \setminus \{(x, y)\})$ is still complete restricted to each triangle as $(x, y) \in \succeq$, and of course if $(\succeq, \succ^*)$ had no triangular cycles then neither could $(\succeq, \succ^* \setminus \{(x, y)\})$. More generally, $(\succeq, \succ^* \setminus Z_0)$ is complete restricted to each triangle in the budget graph, has no triangular cycles, and $\succ \subseteq \succ^* \setminus Z_0$.

Consider now Z_1. If $x \succ^* y$, $x \succeq y$, $y \succeq x$, but $\neg y \succsim_c x$, then there is no reason to include (y, x) in $\succeq$. As $\succ^*$ is asymmetric, we know (y, x) does not belong to $\succ^*$ (as (x, y) does by hypothesis). Moreover, its removal from $\succeq$ cannot affect the completeness of the order pair restricted to each triangle, nor can it create triangular cycles where none previously were. Hence starting from $(\succeq, \succ^*)$, we may instead consider the order pair $(\succeq \setminus Z_1 x, \succ^* \setminus Z_0)$, which has the property that the asymmetric part of $\succeq \setminus Z_1$ is $\succ^* Z_0$, and thus forms a local rationalization for c. $\square$

Henceforth we fix a choice environment (X, Σ), and will suppress the argument

(X, Σ) appearing in domains. Let $\succeq$ be a binary relation on X that is locally rational.[24] Let γ be a loop in $\mathcal{D}$, and $\mathcal{D}|_{\tilde{T}}$ a simple sub-domain of $\mathcal{D}$ containing γ. A 1-form $F \in C^1(\mathcal{D}|_{\tilde{T}})$ is a **cardinalization** of $\succeq$ on $\mathcal{D}|_{\tilde{T}}$ if, for all 1-faces of $\mathcal{D}|_{\tilde{T}}$:

$$y \succeq x \implies F([x, y]) \geq 0,$$

and

$$y \succ x \implies F([x, y]) > 0.$$

Lemma. *(Closed Cardinalization Lemma) Let $\succeq$ be locally rational, and let $\mathcal{D}|_{\tilde{T}} \subseteq \mathcal{D}$ be a simple sub-domain. Then there exists a closed cardinalization of $\succeq$ on $\mathcal{D}|_{\tilde{T}}$.*

Proof. Let $\{\tilde{\tau}_1, \ldots, \tilde{\tau}_I\}$ be an enumeration of those 2-simplices of $\mathcal{D}|_{\tilde{T}}$ that each contain at least two distinct 1-faces in $\dot{\mathcal{D}}|_{\tilde{T}}$. Using this, we construct an enumeration of all 2-simplices of K as follows: between each $\tilde{\tau}_i, \tilde{\tau}_{i+1}$ insert the unique (via combinatorial triviality) ordered sequence of 2-simplices in $\mathcal{D}|_{\tilde{T}}$ connecting them, omitting any 2-simplices of $\mathcal{D}|_{\tilde{T}}$ that have appeared in the construction prior. Let $\{\tau_1, \ldots, \tau_J\}$ denote this enumeration, and let $\mathcal{D}|_{\tilde{T}}^j$ denote the sub-complex of $\mathcal{D}|_{\tilde{T}}$ generated by the first j elements of this enumeration.

We now inductively construct our closed cardinalization of $\succeq$. First, note that there is trivially a closed cardinalization of $\succeq$ on $\mathcal{D}|_{\tilde{T}}^1$: $\succeq$ restricted to the vertices of τ_1 is complete and transitive by local rationality, hence admits a utility function u_1 on these vertices. Let $F_1 \in C^1(\mathcal{D}|_{\tilde{T}}^1)$ be defined as $\mathrm{grad}(u_1)$. For our inductive step, suppose now that there is a closed cardinalization $F_j \in C^1(\mathcal{D}|_{\tilde{T}}^j)$ of $\succeq$ on $\mathcal{D}|_{\tilde{T}}^j$, for some $j < J$. By analogous logic, there is a utility function u_{j+1} representing $\succeq$ restricted to the vertices of τ_{j+1}. Let $\tilde{F}_{j+1} = \mathrm{grad}(u_{j+1})$ be the closed 1-form on the the complex generated by τ_{j+1} alone. By virtue of the structure of the enumeration constructed above, τ_{j+1} and τ_j intersect on exactly a single 1-face, σ with vertex set $\{a, b\}$. There exists some $c \in \mathbb{R}_{++}$ such that $F_j([a, b]) = c\tilde{F}_{j+1}([a, b])$, with c unique if $\succeq$ is strict over this pair.

[24] That is, for all $T \in T_\Gamma$, $\succeq|_T$ is complete and transitive.

Then define:

$$F_{j+1}([x,y]) = \begin{cases} F_j([x,y]) & \text{if } [x,y] \not\subset \tau_{j+1} \\ c\tilde{F}_{j+1}([x,y]) & \text{if } [x,y] \subset \tau_{j+1}, \end{cases}$$

completing the proof. $\square$

Theorem (Ordinal Integrability Theorem)**.** *Let (X,Σ) be a choice environment with $\mathcal{D}(X,\Sigma)$ a simple domain. Then a choice correspondence $c \in \mathcal{C}(X,\Sigma)$ is strongly rationalizable if and only if:*

(i) It obeys the weak axiom; and

(ii) It is locally rationalizable.

Moreover, (i) and (ii) are jointly equivalent to the strong rationalizability of c if and only if $\mathcal{D}(X,\Sigma)$ is simple.

Proof. We begin first by verifying (i) and (ii) are equivalent to strong rationalizability for simple $\mathcal{D}$. Clearly, strong rationalizability always implies (i) and (ii), regardless of the structure of $\mathcal{D}$: any rationalizing weak order $\succeq_c$ of course is a local rationalization and implies $(\succsim_c, \succ_c)$ obeys the weak axiom.

Now, suppose $\mathcal{D}$ is simple, and let $c \in \mathcal{W}(X,\Sigma)$ be locally rationalizable. Let $\gamma \subseteq \mathcal{D}$ be an arbitrary loop. We will show that $\succsim_c|_{E_\gamma}$ cannot be cyclic. As γ is a loop, by simplicity of $\mathcal{D}$ there exists a simple sub-domain $\mathcal{D}|_{\tilde{T}} \subseteq \mathcal{D}$ containing γ, and $\succeq$ a local rationalization of $\succsim_c$ on $\mathcal{D}|_{\tilde{T}}$. By the preceding lemma there exists a closed cardinalization of $\succeq$ on $\mathcal{D}|_{\tilde{T}}$, which we will denote by $F \in C^1(\mathcal{D}|_{\tilde{T}})$. By the cohomology universal coefficient theorem (see Munkres (1984) Theorem 53.1), there exists an isomorphism between $H_1(\mathcal{D}|_{\tilde{T}}, \mathbb{R})$ and $H^1(\mathcal{D}|_{\tilde{T}}, \mathbb{R})$ (see Munkres (1984) Corollary 53.6 or Jiang et al. (2011) Theorem 4), and hence as $\mathcal{D}|_{\tilde{T}}$ is topologically trivial, $H^1(\mathcal{D}|_{\tilde{T}}, \mathbb{R}) = 0$, and therefore there exists an $f \in C^0(\mathcal{D}|_{\tilde{T}})$ such that $\text{grad}(f) = F$. Define the binary relation $\geq^*$ on the vertex set $\mathcal{D}|_{\tilde{T}}^{(0)}$ via $x_0 \geq^* x_1 \iff f(x_0) \geq f(x_1)$ (resp. strict). This is a weak order on the vertices of $\mathcal{D}|_{\tilde{T}}$ which, by consistency of F, is an extension of $\succeq$

on $\mathcal{D}|_{\tilde{T}}$.[25] Thus $\succsim_c |_{E_\gamma}$ is acyclic. As γ was arbitrary, and every potential cycle of $\succsim_c$ must be supported on some loop in $\mathcal{D}$, each contained in some simple sub-domain, we conclude $\succsim_c$ is acyclic. Thus, for all $c \in \mathcal{W}(X, \Sigma)$, if c is also locally rationalizable, it must satisfy the generalized axiom and hence is strongly rationalizable.

We now show that if $\mathcal{D}$ is not simple, (i) and (ii) do not imply strong rationalizability. Suppose, then, that $\mathcal{D}$ is not simple. By Theorem 3 there exists a chordless loop in the budget graph $\Gamma(X, \Sigma)$, which we will denote γ. Thus, there exists a cyclic collection for γ, denoted $\mathcal{B}_\gamma = \{B_1, \dots, B_n\} \subseteq \Sigma$ such that for all $0 \leq j \leq n$ we have $\{x_j, x_{j+1}\} \subseteq B_j$, and that this collection is uncovered: as $|V_\gamma| > 3$ and γ is chordless, no budget in *all of* Σ contains any pair of non-adjacent points in γ. For all $B \in \Sigma|_{\mathcal{B}_\gamma}$, let:

$$\tilde{c}(B) = \begin{cases} e_j \cap e_{j+1} & \text{if } \exists\, e_j \text{ s.t. } e_j = B \cap V_\gamma \\ B \cap V_\gamma & \text{if } |B \cap V_\gamma| = 1 \\ B & \text{else,} \end{cases}$$

and for all $B \in \Sigma$ define:

$$c(B) = \begin{cases} \tilde{c}(B) & \text{if } B \in \Sigma|_{\mathcal{B}_\gamma} \\ B \setminus (\cup_{\tilde{B} \in \mathcal{B}_\gamma} \tilde{B}) & \text{else.} \end{cases}$$

By an argument analogous to that in the proof of Theorem 1, $c \in \mathcal{W}(X, \Sigma)$ and not $\mathcal{G}(X, \Sigma)$.

We now verify that c is nonetheless locally rationalizable. To do this, we will explicitly construct a local rationalization $\succeq$. First, for all $e \in E_\gamma$, let $x_i \prec x_{i+1}$. Thus for all pairs $\{x, y\} \in E_\gamma$, $x \succ y$ if and only if $x \succ_c y$. For all $e \in E_\Gamma \setminus E_\gamma$ that intersect V_γ, we have shown this intersection must be singleton. For all such e, we know e is of the form $\{a, x_i\}$ for some $x_i \in V_\gamma$. For all pairs $\{a, x_i\}$ with $a \in (\cup_{\tilde{B} \in \mathcal{B}_\gamma} \tilde{B})$, define $a \prec x_i$, and if $a \notin (\cup_{\tilde{B} \in \mathcal{B}_\gamma} \tilde{B})$, let $a \succ x_i$. Finally, for those pairs $\{a, b\}$ that do not intersect

[25] It is generally an extension of $\succeq$ (which itself extends the revealed preference $\succsim_c$) as $\geq^*$ is complete and thus generally relates vertices not connected by any edge in $\mathcal{D}|^{(1)}_{\tilde{T}}$.

V_γ, we consider two cases. If, either $\{a,b\} \subseteq (\cup_{\tilde{B}\in\mathcal{B}_\gamma}\tilde{B})$ or $\{a,b\} \subseteq X \setminus (\cup_{\tilde{B}\in\mathcal{B}_\gamma}\tilde{B})$, then let $a \succeq b$ and $b \succeq a$. If exactly exactly one element (without loss a) of $\{a,b\}$ is contained in $(\cup_{\tilde{B}\in\mathcal{B}_\gamma}\tilde{B})$, then let $b \succ a$. Finally, let $\succeq^*$ denote the reflexive closure of $\succeq$. Then $\succeq^* \supseteq \succsim_c$, and, by construction, $\succeq^*$ is locally rational. This follows from (i) for all $\{a,b\} \in E_\Gamma$, either $a \succeq^* b$ or $b \succeq^* a$, and (ii) for every $T \in T_\Gamma$, T only contains at most one pair in γ, as γ is chordless and of length greater than three. Thus, in particular, if T contains an edge of γ, denoted $\{x,y\}$, it contains some element z such that either $z \succ^* x,y$ or $x,y \succ^* z$. If T contains no edges of γ, then $\succeq^* |_T$ is clearly complete and transitive, and hence $\succeq^*$ is locally rational. $\square$

Application Theorems

Proposition. *Let (X,Σ) be a cardinality-constrained choice environment. Then every $c \in \mathcal{W}(X,\Sigma)$ is locally rationalizable if and only if $\mathcal{T}_\Gamma = \Sigma_3$.*

Proof. ($\Longleftarrow$): If $T_\Gamma = \Sigma_3$, consider the revealed preference of any choice function c obeying the weak axiom. If, for any pair $\{x,y\}$ contained in some $T \in \mathcal{T}_\Gamma$, we have neither $x \succsim_c y$ nor $y \succsim_c x$, it means that for every $T' \in \mathcal{T}_\Gamma = \Sigma_3$, it is the case that that neither x nor y are chosen, hence $c(T') = T' \setminus \{x,y\}$. Note that, for any $T \in \Sigma_3$, at most one pair of elements may not have any preference revealed between them, as T is a budget itself so some choice must occur on it. Thus adding both (x,y) and (y,x) to $\succsim_c$ for every such $\succsim_c$-unrelated pair $\{x,y\}$ yields a locally rational extension.

($\Longrightarrow$): We proceed by contraposition. Suppose $\mathcal{T}_\Gamma \neq \Sigma_3$. Of course $\Sigma_3 \subseteq \mathcal{T}_\Gamma$, hence there exists some $\{x,y,z\} \in \mathcal{T}_\Gamma$ that is not a budget itself, but every pair of elements in it is contained in some budget. It is immediate then, due to the cardinality constraints on Σ, that $\{x,y\},\{y,z\},\{z,x\}$ is a loop in $\Gamma(X,\Sigma)$ that possesses an uncovered cyclic collection. By Lemma 2 we obtain the existence of a choice correspondence obeying the weak axiom whose revealed preference exhibits a three-cycle on this loop. Since the vertex set of this loop is in $\mathcal{T}_\Gamma$, this choice correspondence cannot be locally rationalizable. $\square$

Corollary. *Let (X, Σ) be a cardinality-constrained choice environment. Then the weak axiom characterizes strong rationalizability for any choice correspondence if and only (i) $T_\Gamma = \Sigma_3$, and (ii) the domain $\mathcal{D}(X, \Sigma)$ is simple.*

Proof. ($\Longrightarrow$): Suppose $\mathcal{W}(X, \Sigma) = \mathcal{G}(X, \Sigma)$. Then by Theorems 1 and 2, $\mathcal{D}(X, \Sigma)$ is simple and every choice correspondence in $\mathcal{W}(X, \Sigma)$ is locally rationalizable. By the preceding proposition, it follows then that $T_\Gamma = \Sigma_3$.

($\Longleftarrow$): Suppose $T_\Gamma = \Sigma_3$ and $\mathcal{D}(X, \Sigma)$ is simple, and let $\gamma \subseteq \Gamma(X, \Sigma)$ be an arbitrary loop. Then there exists some sub-collection $\tilde{\mathcal{T}}$ of three-good budgets that generate a simple sub-domain containing γ. By the fundamental theorem of simple subdomains, we may take this simple sub-domain's edge set to consist solely of edges of γ and bisections of γ. But, by combinatorial triviality, there exists a 'leaf' triangle in this sub-domain, hence for this triangle there exists a pair of edges $\{x, y\}, \{y, z\} \in E_\gamma$ such that $\{x, y, z\} \in \Sigma_3$. This implies that every cyclic collection for γ is covered, and by the arbitrariness of γ, Σ is well-covered. Theorem 1 then completes the proof. $\square$

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