How Strong is the Weak Axiom?

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Introduction

- For sufficiently rich choice/demand data sets, possible inconsistencies with preference maximization are characterized by simple pairwise, or local consistency conditions.

- This paper is concerned with what constitutes a ‘rich enough’ collection of observations for such results, *independent of the specific choices observed.*

Abstract Choice I

A **choice environment** is a pair (X, Σ), where:

- X is a set of **alternatives**.
- $\Sigma \subseteq 2^X \setminus \{\varnothing\}$ is a collection of non-empty subsets of X called **budgets**.

These budgets correspond to the subsets of X from which we observe the agent choose.

- Assumptions on Σ are assumptions on *observability*.

Abstract Choice II

A **choice correspondence** is a map $c : \Sigma \to 2^X \setminus \{\varnothing\}$ satisfying:

$$(\forall B \in \Sigma) \quad c(B) \subseteq B.$$

The **revealed preference pair** associated to c, denoted $(\succsim_c, \succ_c)$, is defined via:

- $x \succsim_c y$ if there exists a budget $B \in \Sigma$ such that $\{x, y\} \in B$, and $x \in c(B)$.
- $x \succ_c y$ if there exists a budget $B \in \Sigma$ such that $\{x, y\} \in B$, $x \in c(B)$, and $y \notin c(B)$.

Rationalizable Choice

A choice correspondence c is **strongly rationalizable** if there exists a weak order $\succeq$ on X such that:

$$(\forall B \in \Sigma) \quad c(B) = \left\{x \in B : \forall y \in B,\ x \succeq y\right\}$$

The -ARPs

- A choice correspondence satisfies the **weak** axiom of revealed preference (WARP) if it makes no choice *reversals*:

$$x \succsim_c y \implies y \not\succ_c x.$$

- It satisfies the **generalized** axiom of revealed preference (GARP) if it contains no finite choice *cycles* of the form:

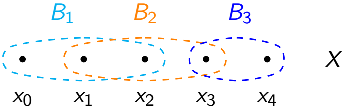
$$x_0 \succsim_c x_1 \succsim_c \cdots \succsim_c x_{N-1} \succ_c x_0.$$

The Budget Graph

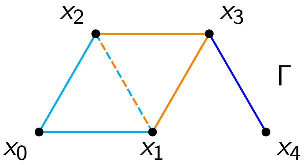
For a choice problem (X, Σ) its **budget graph** Γ is the undirected graph with vertex set $V_\Gamma = X$ and edge set:

$$E_\Gamma = \left\{ \{x, y\} \subseteq X : \exists B \in \Sigma \text{ s.t. } \{x, y\} \subseteq B \right\}.$$

An Example



(a) A choice environment.



(b) The budget graph.

Figure: A choice environment with five alternatives and three budgets.

Cyclic Collections

For a loop $\gamma = (V_\gamma, E_\gamma)$, a collection of budgets $\mathcal{B}_\gamma \subseteq \Sigma$ is a **cyclic collection** for γ if:

$$(\forall e \in E_\gamma)\,(\exists B \in \mathcal{B}_\gamma) \quad e \subseteq B.$$

A cyclic collection $\mathcal{B}_\gamma$ for a loop γ is **covered** if there exists a budget $\bar{B} \subseteq \cup_{\tilde{B} \in \mathcal{B}_\gamma} \tilde{B}$ that either:

(i) Contains V_γ; or

(ii) Contains a pair of elements of V_γ that are not connected by an edge in E_γ.

Note: Condition (i) implies (ii) if and only if $|V_\gamma| > 3$.

Propagation of Choice Cycles

A loop γ has the **propagation property** if every choice correspondence that chooses cyclically around γ necessarily makes another choice cycle elsewhere in the data.

Theorem

A loop in the budget graph has the propagation property if and only if all of its cyclic collections are covered.

Well-covered Budget Collections

A budget collection Σ is **well-covered** if, for every loop γ in its budget graph, every cyclic collection for γ is covered.

Theorem

Let (X, Σ) be a choice environment. The weak axiom of revealed preference is characteristic of strong rationalizability if and only if Σ is well-covered.

Discussion

- Well-coveredness characterizes when choice cycles imply choice reversals.

- This occurs only when *every* loop in the budget graph has the propagation property.

Economic Interpretation of Well-coveredness

- Can we express the idea of well-coveredness in terms of more familiar economic ideas?
- Hurwicz, Uzawa et al. tell us a (nice) demand arises from constrained-optimal choice according to a (nice) utility if and only if its Slutsky matrix is:
  - Negative semi-definite $\iff$ weak axiom; and
  - Symmetric $\iff$ locally integrable.
- *Complete domain*: we know x for every (p, w).

Domains

Let (X, Σ) a be a choice environment with budget graph $\Gamma = (X, E_\Gamma)$.

- The **domain** associated to the choice problem is the triple (X, E_Γ, T_Γ) where:

$$T_\Gamma = \big\{\{x, y, z\} \subseteq X : \{x, y\}, \{y, z\}, \{x, z\} \in E_\Gamma\big\}.$$

- Given some finite collection $\tilde{T} \subseteq T_\Gamma$, the **subdomain** generated by $\tilde{T}$ is the collection of 1-, 2-, and 3-element subsets of elements of $\tilde{T}$.

Local Rationalizability

A choice correspondence is **locally rationalizable** if there exists an order extension $(\succeq, \succ)$ of $(\succsim_c, \succ_c)$ such that:

$$(\forall \tau \in T_\Gamma) \quad \succeq|_\tau \text{ is complete and transitive.}$$

Simple (Sub)domains

A (sub)domain $(\tilde{X}, \tilde{E}, \tilde{T})$ is **simple** if it is:

(i) *Combinatorially Trivial*: If for all $\tau, \tau' \in \tilde{T}$ there is a unique, finite sequence of distinct elements of $\tilde{T}$:

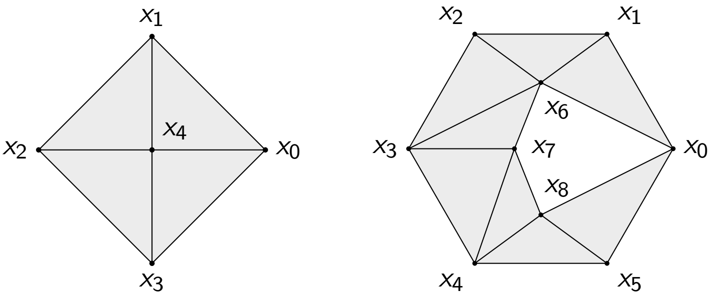
$$\tau = \tau_0, \tau_1, \ldots, \tau_k = \tau'$$

such that τ_j and τ_{j+1} share precisely a pair of elements.

(ii) *Topologically Trivial*: The (sub)domain has a first Betti number of zero.

By abuse of notation, we say a *domain* is simple if every loop in Γ is contained in a simple subdomain.

The Triviality Conditions



(a) A domain that is topologically trivial but not combinatorially trivial.

(b) A domain that is combinatorially trivial but not topologically non-trivial.

Figure: An illustration of the (sub)-domain triviality conditions. Neither domain is simple (i.e. both combinatorially and topologically trivial).

Local versus Strong Rationalizability I

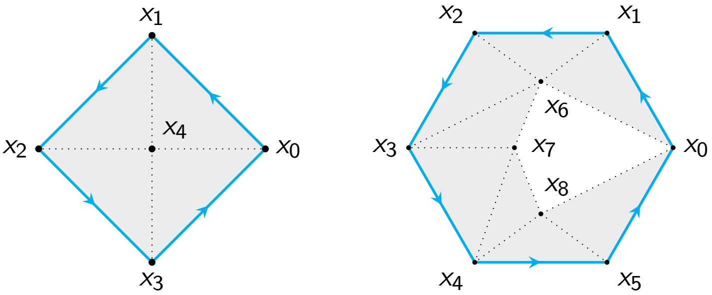


Figure: A cyclic binary relation (blue) on two non-simple domains. Does there exist a locally rational extension?

Local versus Strong Rationalizability II

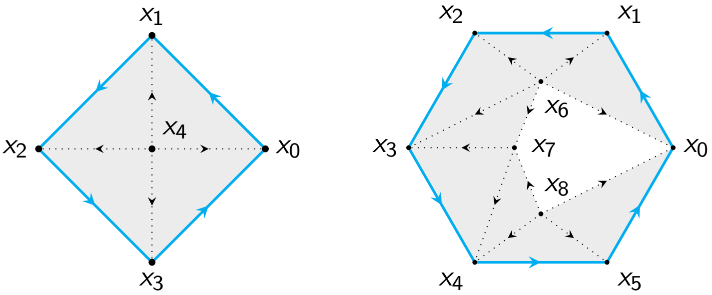


Figure: On non-simple (sub)-domains, locally rational binary relations may not be acyclic.

Ordinal Integrability

Theorem

I. *Let (X, Σ) be a choice environment with a simple domain. Then a choice correspondence is strongly rationalizable if and only if:*

   (i) *It obeys the weak axiom; and*

   (ii) *It is locally rationalizable.*

II. *Moreover, (i) and (ii) are equivalent to strong rationalizability if and only if the domain of (X, Σ) is simple.*

Well-coveredness Revisted

We obtain a decomposition of well-coveredness as: (i) the two classical integrability conditions coincide, and (ii) the choice environment is 'complete enough.'

Theorem

Let (X, Σ) be a choice environment. Then Σ is well-covered if and only if both:

- *If a choice correspondence obeys the weak axiom, then is locally rationalizable; and*
- *The domain for (X, Σ) is simple.*

Application: Inconsistency Indices

- When (X, Σ) is rich, choice cycles can propagate.
- This means not all choice cycles should necessarily be treated independently. Some may be ‘explainable’ by others.
- Measures of irrationality *should account for the structure of the environment*.

Generators for Cycles

- Suppose for some finite (X, Σ) we fix a choice correspondence c. Let $\mathcal{Z}$ denote the set of all choice cycles in the data.
- Given a cycle z of length three or more, any collection of budgets $G_z \subseteq \Sigma$ which **generate** the choice cycle z form a cyclic collection for the loop supporting z.
- We can speak of the budgets that cover the cyclic collection G_z.

A Dependence Relation

- For two choice cycles $z, z' \in \mathcal{Z}$, we say z **induces** z', denoted $z \to z'$, if there exist generators for the cycles $G_z, G_{z'} \subseteq \Sigma$ such that:

$$G_{z'} \subseteq G_z \cup \{B \in \Sigma : B \text{ covers } G_z\}.$$

- Given only those choices made on budgets in G_z, every choice on a cover for G_z necessarily forces another cycle. We shouldn't deem any induced cycles as representative of a deeper degree of irrationality.

The Irrational Kernel

Consider the transitive closure of $\to$ on $\mathcal{Z}$, denoted $\to^*$. We call a collection $\mathcal{I} \subseteq \mathcal{Z}$ an **irrational kernel** for the data if:

(i) For all $z' \in \mathcal{Z}$ there exists an $z \in \mathcal{I}$ such that:

$$z \to^* z'.$$

(ii) $\mathcal{I}$ is minimal amongst all such collections of cycles.

Inconsistency Indices

- The size of any irrational kernel gives an integer-valued measure of how many ‘independent’ cycles are in a data set.
- Particularly, it accounts for how the structure of the environment affects the number of cycles.
- Generally, data will be richer than just the abstract choice framework. This extra data should of course be used. But any reasonable inconsistency index should be increasing in the size of the irrational kernel.

Conclusions

- The manner in which the structure of the choice environment affects and constrains the possible revealed preference data is not well understood.
- We characterize how the environment may cause cycles in choice data to induce other cycles elsewhere in the data.
- This property turns out to be intimately related to the richness condition characterizing when the weak and generalized axioms coincide, as well as to understanding when analogues of classical integrability theory obtain under incomplete data.
- These results also yield a simple manner to appropriately 'count' GARP violations in data for experimentalists.

Thank you!

Any Questions?