

Interpersonally Comparable Utility*

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Abstract

We develop a theory of measurement scales for utility functions, based on a generalization of the notion of numeraire commodity. Every sufficiently well-behaved utility is denominated in some scale of this form, and conversely, the choice of a compatible scale unit uniquely identifies utilities up to an additive constant. We define a profile of utilities to be interpersonally comparable precisely when they are denominated in a common measurement unit. We study when profiles of comparable utilities exist, as well as how a planner ought to aggregate them, and provide applications to social choice and welfare economics.

“If we want to compare utilities between two people much the same problem exists . . . we do not seem to have any ‘outside thing’ which can be measured by both persons to ascertain the relation between the units.”
—Luce and Raiffa (1957) p. 34.

1 Introduction

The question of how to obtain cardinally meaningful, and interpersonally comparable, measures of individual well-being is foundational to much of modern economics.¹

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¹For example, cost-benefit analysis weighs the gains of some individuals against the losses of others (Chipman and Moore 1976, 1980; Schlee and Khan 2023), optimal taxation considers reform

However, despite the centrality of these notions, they have frustrated nearly all attempts to give them concrete, formal meaning within the standard framework of economic theory.² This has given rise to a broad consensus that any methodology for systematically considering such comparisons must necessarily make use of normative, external judgments.³

The objective of this paper is to develop a theory of comparable utilities, relying only on ordinal preference information.⁴ Rather than emphasize any particular choice of representation, we build on work from [Caradonna \(2024\)](#) to introduce a notion of a ‘measurement unit’ for utilities. This allows us to formally speak of which possible units a given utility function is denominated in, without constraining the values it assigns to particular outcomes.

This, in turn, provides a natural criterion for interpersonal comparability. We define a profile of utilities to be interpersonally comparable precisely when there exists a common unit in which every representation is denominated. Our main results characterize when a profile of utilities admits such a common unit, supply axiomatic foundations for the utilitarian aggregation of comparable utilities, and describe the jointly comparable representations of profiles of ordinal preferences.

Critically, this is not a positive theory of utility—we make no claim that comparable profiles reflect the literal felicities experienced by each individual. Instead, our results provide a benchmark for when it is justifiable, or *rationalizable*, for a planner to treat utility differences, across both pairs of outcomes and individuals, as quantified on a common scale. We illustrate this below.

Example 1. Suppose society consists of S individuals with preferences over bundles of two commodities, $x = (x^1, x^2) \gg 0$, that can be represented by a quasilinear utility function:

$$u^s(x^1, x^2) = v^s(x^2) + x^1. \quad (1)$$

incidence across households ([Guesnerie 1977](#); [Tirole and Guesnerie 1981](#); [Saez and Stantcheva 2016](#); [Sher 2024](#)); social choice compares both utility levels, and differences, across individuals ([Sen 1970, 1974](#); [d’Aspremont and Gevers 1977](#); [Roberts 1980a,b](#)).

²For an excellent overview, see [Hammond \(1991\)](#).

³E.g. [Robbins \(1938\)](#); [Arrow \(1950\)](#); [Samuelson \(1954\)](#); [Sen \(1999\)](#). See also [Fleurbaey and Hammond \(2004\)](#) for further discussion.

⁴That is, on the basis of ordinal preferences defined over pure outcomes. In particular, we do not consider ‘extended’ preferences over outcome-individual pairs, e.g. [Suzumura \(1996\)](#).

Let $\varphi_t(x) = x + (t, 0)$ denote the transformations which add t units of the numeraire commodity, whenever feasible.⁵ Such transformations are objectively meaningful, as they are defined without reference to any preference. Moreover, every utility (1) satisfies the functional equations:

$$u^s(\varphi_t(x)) = u^s(x) + t, \quad (2)$$

for every bundle x and feasible quantity t . Thus, for every individual, a utility difference of t has the same interpretation: the value of t additional units of the horizontal-axis commodity. This common scale is not obtained by prescribing particular utility levels to specific allocations. Instead, it is supplied by the exogenously-defined family of transformations $\{\varphi_t\}$. In such circumstances, we regard the utilities u^s as interpersonally comparable. ■

By relying on general transfer functions, rather than physical commodities, we are able to define numeraire-like structure more broadly. When society's preferences are over a space of social states or allocations $\mathcal{X}$, we define a 'virtual' commodity simply as a collection of transfer functions, φ_t , each mapping $\mathcal{X} \rightarrow \mathcal{X}$, satisfying:

$$\varphi_0(x) = x \quad \text{and} \quad \varphi_{t'}(\varphi_t(x)) = \varphi_{t'+t}(x),$$

whenever the relevant transfers are feasible.⁶ We interpret each φ_t as augmenting a given state, or allocation, with t additional units of an abstract good. Such a commodity defines a utility unit for a representation u whenever:

$$u(\varphi_t(x)) = u(x) + t,$$

for every feasible x and t , and we define a profile $(u^1, \dots, u^S)$ to be interpersonally comparable when it admits a common utility unit. Because the transforms φ_t are defined over social states, without reference to any preference or utility, they provide a common, objective scale of measurement. In the language of [Luce and Raiffa \(1957\)](#), they supply precisely the "outside thing," against which each utility can be measured, to ascertain the equivalence of their units.

Our first main result shows that monotonicity, and a simple 'no wealth effects' axiom, characterize the set of virtual commodities that are utility units for some

⁵Formally, when t is negative, we require that $(x^1 + t, x^2)$ remain component-wise positive.

⁶More generally, we allow $\varphi_t(x)$ to not be defined for all x, t pairs, to reflect that certain transfers may not be feasible given bounds or other constraints on the space of states $\mathcal{X}$.

representation of a given preference. Conversely, we show *every* utility representation of a preference admits a unit of this form, which uniquely determines it, up to an additive constant. Thus, the selection of a valid commodity of this form is equivalent to the selection of a privileged measurement unit, or “util,” in which to numerically denominate a representation.

This also highlights that our notion of comparability imposes material restrictions on profiles of utilities. If a profile contains two utilities for the same preference, that differ by a nonlinear, monotone transform, no common utility unit can exist. More generally, we show a simple geometric condition on the gradients of a utility profile completely characterizes comparability, providing a straightforward test for verifying it in practical applications.

We then consider how a social planner ought to aggregate such utilities. Restricted to the domain of comparable profiles, we show a planner who uses the cardinal measurements, and interpersonal comparisons, justified on the basis of a common utility unit, but who otherwise satisfies the axioms of [Arrow \(1951\)](#), is necessarily utilitarian. Crucially, this characterization endogenously justifies the cardinal and interpersonal comparisons it relies on, rather than assuming the informational basis for making them is externally given. Thus, our notion of comparability provides an underlying, theoretical basis for the *unit comparability* assumptions widely used in social choice, e.g. [Sen \(1970\)](#); [d’Aspremont and Gevers \(1977\)](#); [Roberts \(1980a,b\)](#); [Bossert and Weymark \(2004\)](#).

Finally, we consider the problem of whether a tuple of ordinal preferences admits any comparable profile of representations, and if so, how many. When multiple comparable profiles exist, each corresponds to a different choice of utility unit in which to measure well-being. Beyond yielding differing normalizations for individual utilities, the associated utilitarian social preference obtained from each comparable profile will generally differ. In such cases, additional economic or normative considerations are required to select from the set of admissible units.⁷

For profiles of expected-utility preferences, this multiplicity provides a unifying

⁷However, even in such cases, the requirement of comparability starkly restricts the set of utilitarian social preferences. For any sufficiently rich tuple of preferences, we show the collection of comparable profiles is always finite-dimensional, and thus makes up a vanishing proportion of the set of possible profiles of representations. See [Theorem 4](#) and [Theorem 6](#).

perspective on several well-known theories. We show utilitarian aggregation of von Neumann-Morgenstern utilities à la [Harsanyi \(1953, 1955\)](#), relative utilitarianism ([Dhillon and Mertens 1999](#)), and the Nash social welfare function ([Kaneko and Nakamura 1979](#)), all correspond to utilitarian aggregation of comparable profiles, differing only in the choice of common measurement unit. Moreover, for sufficiently rich profiles, we prove *every* interpersonally-valid utilitarian social preference takes one of these three forms, or corresponds to a simple, parametric combination of them.

In particular, we provide new justification for methods of aggregation that rely on profiles of von Neumann-Morgenstern utilities, or their normalized logarithms. Apart from their convenience and interpretational simplicity, we show such representations make up essentially the only interpersonally comparable representations for expected utility preferences. This provides a new perspective on a longstanding critique of Harsanyi’s theorems, which asserts his reliance on expected utilities is unjustified. See, e.g., [Weymark \(2005\)](#); [Roemer \(2008\)](#). Similar critiques have also been raised for the Nash social welfare function ([Gonczarowski and Segev 2024](#)).

In general, describing the set of comparable profiles of representations (or even determining its non-emptiness), for given preferences, is a difficult problem. However, in the setting of local welfare analysis (e.g. [Guesnerie 1977](#); [Milleron 1970](#); [Arrow and Lind 1970](#); [Radner 1993](#)), we provide a complete characterization. Our results rely on a suite of results due to [Cartan \(1908\)](#), developed as part of Felix Klein’s *Erlangen* program ([Klein 1872](#)), which are collectively known as Cartan’s ‘method of equivalence.’ These techniques have found widespread application in physics, geometry, optimization, and control theory.⁸ Nevertheless, due to its more technical nature, we treat this question in the [Supplemental Appendix](#).

Finally, we illustrate that our notion of comparability yields new insights in problems of applied welfare analysis. We consider a planner selecting between marginal indirect tax reforms when households’ direct utilities are comparable. By explicit example, we show that even in simple settings, utilitarian aggregation of comparable utilities can yield the opposite ranking of two revenue-equivalent reforms relative to the sum of money metric utilities. The discrepancy arises as the money metric utilities may fail to be comparable, even when constructed using the common, reference

⁸See, e.g., [Karlhede \(1980\)](#); [Gardner \(1989\)](#); [Olver \(1995\)](#); [Gallo et al. \(2005\)](#) and references therein.

prices.

2 Related Literature

The question of how to obtain cardinally meaningful, and interpersonally comparable, utilities is the subject of longstanding literatures, both in economics and philosophy. For excellent overviews, see [Hammond \(1991\)](#) or [Fleurbaey and Hammond \(2004\)](#), and references within.

Social Choice: [Arrow \(1951\)](#) explicitly argues against the possibility of cardinal, interpersonally comparable utility, leading to his impossibility theorem. This assumption was later revisited ([Sen 1970](#), [Sen 1974](#), [d’Aspremont and Gevers 1977](#), [Roberts 1980a,b](#), [Blackorby et al. 1984](#)), who obtain possibility results when various types of interpersonal comparability are exogenously assumed.⁹ A common critique of this approach is that it is unclear when such comparisons are justified ([Fleurbaey and Hammond 2004](#)).¹⁰ While these works study the planner’s problem when presented with cardinal, interpersonally comparable utilities, it has remained an open question to establish when, and how, such comparisons are justified. Our contribution is to address precisely this gap.

A common approach to motivating interpersonal comparability is to restrict preferences to quasilinear or expected utility form (e.g. [Harsanyi 1955](#); [Moulin 1985](#); [Gonczarowski and Segev 2024](#)). From this perspective, our notion of a utility unit may be seen as generalizing quasilinearity to a far wider range of profiles and settings. In the [Supplemental Appendix](#), we show comparability is equivalent to the existence of a global change-of-coordinates which renders a profile simultaneously quasilinear, with respect to a common numeraire.¹¹

Classical welfare economics often sidestepped matters of comparability by assuming that all consumers share a common underlying preference, possibly differing

⁹See also, e.g., [DeMeyer and Plott \(1971\)](#) for similar ideas in the context of voting, and [Bossert and Weymark \(2004\)](#) for an excellent overview of these ideas.

¹⁰[Fleurbaey and Hammond \(2004\)](#) (p.1271) write “In retrospect, however, the social welfare functional approach can now be seen to have several quite serious defects. One is the failure to explain how interpersonal comparisons of utility are to be interpreted, since the informational basis was assumed to be exogenously given.”

¹¹See [Theorem 5](#).

only in their specific representation (Vickrey 1945; Mirrlees 1971). More modern approaches include the use of equivalence scales (e.g. Deaton and Muellbauer 1980; Lewbel 1989) or consideration only of preferences whose behavior admits an exact aggregate, e.g. Gorman (1953).¹² Because we allow for nonlinearities in our generalized numeraire, comparable profiles generally need not have demand behavior consistent with any representative consumer. Instead, comparability allows one to directly work with the utilitarian social preference obtained from the sum of the individual utilities.

Welfare Economics: Various approaches exist to constructing normalized utility functions with particular units of measure. Wold (1944) provides a construction of a continuous utility function in commodity space which is normalized to be linear along a particular line segment. Similar ideas appear in the literature on ‘distance functions’ (Malmquist 1953; Deaton 1979).¹³ The practical value of these approaches is discussed in Deaton (1980).

A widely used method in consumer theory relies on the use of reference prices to obtain money metric representations (McKenzie 1957; Samuelson 1974). Chipman and Moore (1976); Willig (1976); Chipman and Moore (1980, 1992) study the cardinal meaningfulness of such metrics in the setting of cost-benefit analysis. While empirically and theoretically attractive (e.g. Deaton 1980; Deaton and Zaidi 2002; Schlee and Khan 2023), there is limited justification for treating money metric utilities as interpersonally comparable (Hammond 1991, 1994).¹⁴

Extensive Measurement: The theory of extensive measurement (Suppes 1951; Krantz et al. 1971) is perhaps closest to our approach. It seeks to obtain canonical, cardinally meaningful numerical representations for orderings, by leveraging algebraic structure on the space of objects over which measurements are constructed. These ideas have found recent application in social welfare (Nebel 2023, 2024). Our approach may be viewed as somewhat dual. Rather than rely on exogenous algebraic structure on social states to construct a measurement scale, we study when a particular profile of preferences admits sufficient, common algebraic structure for extensive measurement to be carried out.

¹²For an overview of this approach see Section 6 of Fleurbaey and Hammond (2004).

¹³Debreu (1951) suggests a related measure for quantifying inefficiency.

¹⁴Indeed, Samuelson (1974) (p. 1266) calls the utilitarian summation of money metrics for welfare analysis a ‘perversion,’ and explicitly cautions ‘any such temptation should be resisted.’

Geometry of Preferences & Preference Profiles: Formally, our notion of a virtual numeraire is a particular type of continuous symmetry of a preference (Tyson 2013; Mantovi 2016). Ok and Riella (2014, 2021) consider symmetries of incomplete preferences defined on groups, and study their extension properties. Caradonna and Chambers (2026) highlight the ubiquity of such symmetries in axiomatic work, and develop a general revealed preference theory for testing them.

An earlier notion of a virtual numeraire was proposed by Caradonna (2024), in the context of revealed preference theory. Our definition extends the one given there by allowing for transfers of sufficient magnitudes to be infeasible, providing a more flexible theory.¹⁵

Our main results concern the simultaneous restrictions imposed by preference *profiles*, rendering them distinct from existing work studying the geometry of individual preferences (Debreu 1959, 1972; Mas-Colell 1985; Chiappori and Ekeland 1999, 2006). Recently, Sandomirskiy and Ushchev (2024) apply techniques from convex geometry to study the aggregation properties of profiles of homothetic preferences.

3 Measurement of Individual Utility

Let $\mathcal{X}$ be a d -dimensional, convex, relatively open subset of Euclidean space, representing the possible social states, or **allocations** available to society. We will assume $d \geq 2$. The set of individuals comprising the society in question is denoted $\mathcal{S} = \{1, \dots, S\}$.

A **utility function** is a smooth, critical-point-free map $u : \mathcal{X} \rightarrow \mathbb{R}$. Similarly, a **preference relation** $\succsim$ is a complete and transitive binary relation on $\mathcal{X}$ such that, for some utility function u :

$$x \succsim y \iff u(x) \geq u(y),$$

¹⁵However, the theory developed in Caradonna (2024) allows for non-smooth virtual commodities and is valid in any metric consumption space. Here, we focus on the finite-dimensional smooth theory to allow for the use of geometric techniques to establish existence results. Part (i) of our Theorem 1 is also similar to Theorem 1 of Caradonna (2024). However, its proof relies on different underlying arguments, and it provides a strictly smaller axiom set, relative to Caradonna (2024), precisely due to this additional analytic structure.

for all $x, y \in \mathcal{X}$. We will use $\succ$ (resp. $\sim$) to denote the asymmetric (resp. symmetric) components of $\succsim$.

We allow allocations in $\mathcal{X}$ to be either public or private. Moreover, when allocations are private, we explicitly allow individuals' preferences to depend not only on their own components of the allocation, but also on those of other members of society.

3.1 Virtual Commodities

We now introduce our core technical construction, virtual commodities. Informally, a virtual commodity consists of a family of transformations, each mapping $\mathcal{X} \rightarrow \mathcal{X}$. We interpret each function as augmenting an allocation with some fixed quantity of a nonlinear composite commodity. However, as $\mathcal{X}$ itself may be bounded, not every allocation can be endowed with arbitrarily large quantities of additional commodity.

A **commodity domain** is an open set $\mathcal{D} \subseteq \mathcal{X} \times \mathbb{R}$ such that, for each $x \in \mathcal{X}$, the set:

$$\mathcal{D}^{(x)} = \{t : (x, t) \in \mathcal{D}\}$$

is an open interval containing zero. A smooth map $\varphi : \mathcal{D} \rightarrow \mathcal{X}$ is a **transfer function** if, writing $\varphi_t(x)$ for $\varphi(x, t)$, for all $x \in \mathcal{X}$ both (i) $\varphi_0(x) = x$, and (ii) $t_1 \in \mathcal{D}^{(x)}$ and $t_2 \in \mathcal{D}^{(\varphi_{t_1}(x))}$ implies $\varphi_{t_1+t_2}(x)$ is well-defined, and $\varphi_{t_2}(\varphi_{t_1}(x)) = \varphi_{t_1+t_2}(x)$.¹⁶

A **virtual commodity** is a pair $(\mathcal{D}, \varphi)$, where $\mathcal{D}$ is a commodity domain, and φ a transfer function, which is maximal.¹⁷ For any $t \in \mathcal{D}^{(x)}$, we interpret the allocation $\varphi_t(x)$ as x plus t units of the virtual commodity; the set $\mathcal{D}^{(x)}$ reflects the amounts of commodity that can be feasibly added to, or subtracted from, x . Thus, condition (i) simply requires that adding no units of commodity not affect any allocation, while (ii) is a mild, path-independence property, stating that adding t_2 units of commodity, to the allocation consisting of x plus t_1 units, is equivalent to adding $t_1 + t_2$ units to x all at once.

Finally, given any virtual commodity φ , its **generator** is the vector field:

$$\dot{\varphi}(x) = \frac{\partial}{\partial t} \varphi_t(x) \Big|_{t=0}.$$

¹⁶By well-defined, we mean that $(t_1 + t_2) \in \mathcal{D}^{(x)}$.

¹⁷That is, there exists no commodity-domain/transfer-function pair $(\mathcal{D}', \varphi')$ with $\mathcal{D} \subset \mathcal{D}'$ and $\varphi = \varphi'|_{\mathcal{D}}$.

The vector $\dot{\varphi}(x)$ describes the perturbation at $x \in \mathcal{X}$ corresponding to adding an infinitesimal additional quantity of commodity.

Example 2. Suppose $\mathcal{X} = \mathbb{R}_{++}^d$ consists of all bundles of positive quantities of d consumption goods. Any of these d goods may be expressed as a virtual commodity. For example, the map:

$$\varphi_t(x) = x + (t, 0, \dots, 0)$$

represents the horizontal-axis good as a virtual commodity.¹⁸ Here, applying the transformation φ_t to some bundle x , augments it with t additional units of commodity one, and the generator $\dot{\varphi}(x) = [1, 0, \dots, 0]^\top$ for all $x \in \mathcal{X}$.

However, virtual commodities are significantly more flexible, and allow for non-linear transfers. For example, $\varphi_t(x) = e^t x$, with $\mathcal{D} = \mathbb{R}_{++}^d \times \mathbb{R}$, defines a virtual commodity, where $\dot{\varphi}(x) = x$. Here, ‘adding’ t extra units of commodity instead proportionally scales the bundle x by the scalar e^t . ■

3.2 Measurement of Individual Utility

A virtual commodity $(\mathcal{D}, \varphi)$ is a **virtual numeraire** for a preference $\succsim$ if it satisfies the following pair of axioms.

(N.1) **Invariance:** For all $x, y \in \mathcal{X}$, and $t \in \mathcal{D}^{(x)} \cap \mathcal{D}^{(y)}$:

$$x \succsim y \iff \varphi_t(x) \succsim \varphi_t(y).$$

(N.2) **Monotonicity:** For all $x \in \mathcal{X}$, and $t \in \mathcal{D}^{(x)}$:

$$\varphi_t(x) \underset{(>)}{\succsim} x \iff t \underset{(>)}{\geq} 0.$$

Invariance (N.1) says that adding some common quantity of numeraire to two allocations does not affect the preference between them. It rules out cases in which adding the same quantity of commodity to each causes the preference between them to reverse. Monotonicity (N.2) says the virtual commodity is a good.

¹⁸The relevant commodity domain is defined by $\mathcal{D}^{(x)} = (-x^1, \infty)$.

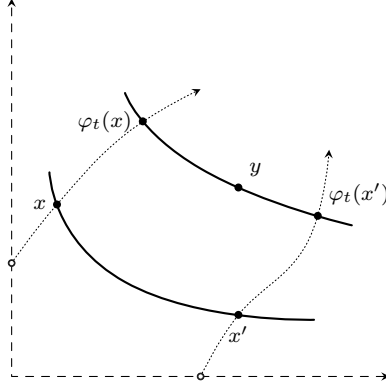


Figure 1: Two sequences of the form (4), of length one. The virtual numeraire φ provides a measurement scale for quantifying the intensity of the preference $y \succ x$. A utility is denominated in φ units whenever the total transfer, along any such sequence, equals the utility difference.

Suppose that $(\mathcal{D}, \varphi)$ is a virtual numeraire for a preference $\succsim$, which is represented by some utility function u . We say that $(\mathcal{D}, \varphi)$ is a **utility unit** for u if, for all $(x, t) \in \mathcal{D}$:¹⁹

$$u(\varphi_t(x)) = u(x) + t. \quad (3)$$

In such cases, willingness-to-pay, denominated in φ terms, provides an exact proxy for utility differences. For example, if for any x and y , there exists $x' \sim x$, $t \in \mathcal{D}^{(x')}$, such that $y \sim \varphi_t(x')$, then by (3) it must be that $t = u(y) - u(x)$. More generally, as our next proposition shows, every x and y are related by a finite sequence of transfers of this form, whose sum uniquely determines $u(y) - u(x)$.

Proposition 1. *Let $(\mathcal{D}, \varphi)$ be a virtual numeraire for $\succsim$. Then for any $x, y \in \mathcal{X}$, there exists a finite sequence $(x_1, t_1), \dots, (x_K, t_K) \in \mathcal{D}$ such that:*

$$x \sim x_1, \quad \varphi_{t_1}(x_1) \sim x_2, \quad \dots \quad \varphi_{t_{K-1}}(x_{K-1}) \sim x_K, \quad \varphi_{t_K}(x_K) \sim y. \quad (4)$$

Moreover, if $(\mathcal{D}, \varphi)$ is a utility unit for some representation u , then for any such sequence, $u(y) - u(x) = \sum_{k=1}^K t_k$.

Thus, when φ is a utility unit for u , we may regard each transformation φ_t as a ‘ t -util’ measuring stick; see Figure 1. By Proposition 1, these allow for rich enough measurements to capture the utility difference between any pair of allocations.

¹⁹Our restriction to virtual numeraires here is without loss of generality. If a virtual commodity satisfies (3), it necessarily satisfies (N.1) and (N.2) for the underlying preference.

Our first main result establishes two basic facts. The first is that every virtual numeraire for a preference on $\mathcal{X}$ is a utility unit for some choice of representation. Thus the collection of relative intensity measurements, captured by the virtual numeraires for a given preference, is no richer than the set of utility differences generated by its possible representations.

The second is that, conversely, for any preference $\succsim$ and choice of representation u , there exists a virtual numeraire for $\succsim$ which satisfies (3) for u . Thus, virtual numeraires yield a sufficiently rich theory of measurement to capture the utility differences arising from any choice of representation, for any preference. Together, these imply the selection of some virtual numeraire for a preference may be regarded precisely as a choice of ‘util,’ or measurement unit for utility.

Theorem 1. *Let $\succsim$ be a preference on $\mathcal{X}$. Then:*

- (i) *Every virtual numeraire for $\succsim$ is a utility unit for some choice of representation.*
- (ii) *For any utility representation of $\succsim$, there exists a virtual numeraire which is a utility unit.*

Moreover, two representations of $\succsim$ share a common utility unit if and only if they differ by an additive constant.

Theorem 1 shows virtual numeraires satisfy the basic desiderata for any notion of cardinal utility unit. For any preference, fixing a choice of virtual numeraire identifies a unique (up to an additive constant) representation satisfying (3). In this sense, such commodities precisely capture the *measurement unit* for utility, rather than speaking to any means of quantifying utility level (Krantz et al. 1971).

Critically, virtual commodities may be defined without reference to any preference. In this sense they are *objective* units of measure, external to the ordinal (or cardinal) assessments made by any individual. Thus, rather than thinking of the property of being numeraire for a given preference as an attribute of a commodity, it is perhaps more accurate to view the commodity as primitive, and (N.1) and (N.2) as characterizing when a preference is suitably ‘measurable’ with respect to it.

Example 3. Consider a finite set of prizes, $\mathcal{Z}$, partitioned into two sets, G and B , reflecting good and bad outcomes. Individuals are assumed to have preferences

defined on $\mathcal{X} = \text{int } \Delta(\mathcal{Z})$, i.e. full-support lotteries over these prizes, which can be represented by:

$$u^s(p) = \ln \left(\frac{p(G)}{p(B)} \right) + v^s(p_G, p_B),$$

where $p(G)$ and $p(B)$ denote the probabilities of the events G and B under p , and p_G and p_B the probabilities conditional on these events. Thus each utility u^s is the sum of the log-odds of receiving a non-disappointing prize, plus an arbitrary function of the conditional probabilities, p_G and p_B .

Despite there not existing any physical commodities in this environment, the virtual commodity:

$$\varphi_t^j(p) = \frac{e^{t \cdot \mathbf{1}_{j \in G}} p^j}{p(B) + e^t p(G)}, \quad j \in G \cup B,$$

defines a common virtual numeraire. Here, ‘adding’ additional commodity shifts mass away from the disappointing prizes in B , to the more desirable outcomes in G , while preserving the conditional probabilities p_G and p_B . In particular, every u^s satisfies (3), and hence treats φ as a common utility unit. ■

An important caveat is that any fixed utility representation admits infinitely many possible units. For example, when $\mathcal{X} = \mathbb{R}^2$ and $u(x^1, x^2) = x^1 + x^2$, any vector $\lambda = (\lambda^1, \lambda^2)$ with $\lambda^1 + \lambda^2 = 1$ defines a utility unit for u , via:

$$\mathcal{D} = \mathbb{R}^3 \quad \text{and} \quad \varphi_t(x) = x + t\lambda.$$

Thus while [Theorem 1](#) tells us that fixing a choice of representation dramatically reduces the possible utility units, it does not provide exact identification.²⁰

4 Interpersonally Comparable Utility

Suppose now that $(u^1, \dots, u^S)$ is a profile of utilities. We will say these utilities are **interpersonally comparable** if there exists a common utility unit, i.e. a virtual commodity $(\mathcal{D}, \varphi)$ such that, for all $s \in \mathcal{S}$ and all $(x, t) \in \mathcal{D}$,

$$u^s(\varphi_t(x)) = u^s(x) + t.$$

²⁰However, by [Theorem 1](#), a virtual numeraire can be a utility unit of two representations u and v of some $\succsim$ if and only if $u = v + c$ for some constant c . Thus fixing u as a choice of representation rules out those virtual numeraires which satisfy (3) for any nonlinear (indeed, non-translational) monotone transform of u .

When such a commodity exists, each utility u^s may be regarded as being denominated in the same objective measurement unit, the numeraire good φ .

We interpret the property of being interpersonally comparable as indicating that it is *rationalizable*, or internally consistent, for a social planner or policymaker to treat the profile of utilities as denominated in a common measurement unit. This provides justification for comparing the magnitude of the gains of some members of society against the losses of others, as captured by these utilities.

An immediate consequence of this definition is that not all utilities are interpersonally comparable. For example, by [Theorem 1](#), if society contains two individuals with the same underlying preference, but whose utilities differ by a nonlinear monotone transform, the resulting profile cannot be interpersonally comparable. Intuitively, such a pair of utilities cannot be rationalized as being denominated in any common measurement unit.

Interpersonal comparability also yields ordinal implications.^{[21](#)} If society's underlying preferences are too heterogeneous, interpersonal comparability can fail for *every* profile of representations. For example, if at some allocation x , the intersection of all individuals' strictly preferred sets $\{x' \in \mathcal{X} : x' \succ^s x\}$ is empty, no virtual commodity can satisfy [\(N.2\)](#) for every preference, and hence no common virtual numeraire can exist.^{[22](#)} Thus, given sufficiently detailed information about subjects' preferences (or utilities), the hypothesis that a profile is interpersonally comparable is falsifiable.^{[23](#)}

Finally, we note that comparability is a cardinal property. When a profile $(u^1, \dots, u^S)$ admits a common unit, even if an identical, strictly increasing transform is applied to each utility, the resulting profile will generally not be interpersonally comparable. The following proposition characterizes the tuples of monotone transforms which robustly preserve comparability.

Proposition 2. *Suppose $S \geq 2$, and let $f = (f^1, \dots, f^S)$ be a tuple of smooth, strictly increasing, and critical-point-free transformations. Then f has the property that $(f^1 \circ u^1, \dots, f^S \circ u^S)$ are interpersonally comparable whenever $(u^1, \dots, u^S)$ are,*

²¹See also [Appendix H](#).

²²This is closely related to various domain and profile restrictions in social choice, see [Kramer \(1973\)](#); [Chichilnisky and Heal \(1983\)](#); [Dhillon and Mertens \(1999\)](#).

²³In the case of a preference profile, we take feasibility to mean the existence of some profile of representations that is jointly comparable.

if, and only if:

$$f^s(t) = \alpha t + \beta^s, \quad (5)$$

for $\alpha \in \mathbb{R}_{++}$ and $\beta^s \in \mathbb{R}$, for all $s \in \mathcal{S}$.

4.1 Comparability & Arrovian Aggregation

How ought a planner synthesize a profile of comparable utilities into an aggregate, social ranking? Relative to much of social choice, which supposes an unrestricted domain (see, e.g. [Bossert and Weymark 2004](#)), as noted above, comparability restricts the set of allowable profiles.

Let $\mathcal{U}$ denote all profiles of utilities $(u^1, \dots, u^S)$ on $\mathcal{X}$, and $\mathcal{C} \subset \mathcal{U}$ the sub-collection of comparable profiles. A social welfare functional F is a mapping from $\mathcal{U}$ to $\mathcal{P}$, the set of preference relations on $\mathcal{X}$. For fixed F and any $u \in \mathcal{U}$, we denote the preference $F(u)$ via $\succeq_u$, with asymmetric component $\succ_u$.

Following [Sen \(1970\)](#), a social welfare functional is able to rely on cardinal information about utility differences, both intra- and interpersonally, though not information about utility levels, when it satisfies the following axiom.²⁴

(CUC) **Cardinal Unit-Comparability:** For any profile $(u^1, \dots, u^S) \in \mathcal{U}$, and any $\alpha > 0, \beta \in \mathbb{R}^S$:

$$F(u^1, \dots, u^S) = F(\alpha u^1 + \beta^1, \dots, \alpha u^S + \beta^S).$$

By definition, if $u \in \mathcal{C}$, it is meaningful to make comparisons of the form:

$$u^s(x) - u^s(y) \geq u^{s'}(x') - u^{s'}(y'), \quad (6)$$

for any $s, s' \in \mathcal{S}$ and any $x, x', y, y' \in \mathcal{X}$, as by [Proposition 1](#), the quantities on each side of the inequality can be expressed in terms of the common utility unit, φ . Conversely, (CUC) says that when a profile of utilities is scaled by a common factor

²⁴In contrast, the case of [Arrow \(1951\)](#), where F is able to only rely on ordinal information, can be captured by requiring that for any $(u^1, \dots, u^S) \in \mathcal{U}$, and any tuple $(f^1, \dots, f^S)$ of increasing transforms:

$$F(f^1 \circ u^1, \dots, f^S \circ u^S) = F(u^1, \dots, u^S).$$

$\alpha > 0$, or shifted by individual quantities β^s , the induced (ordinal) ranking over social alternatives must remain unchanged. By Bossert (1991), such transformations are precisely those which preserve the inequalities (6). Thus, for a social welfare functional defined over comparable profiles, (CUC) simply encodes the requirement that the planner use only those measurements which can be constructed using φ , i.e. (6), and nothing more.

Apart from (CUC), we consider a planner who is otherwise Arrovian, that is, who satisfies the social welfare functional analogues of the axioms of Arrow (1951). Our treatment here follows d’Aspremont and Gevers (1977).

- (IIA) **Independence of Irrelevant Alternatives:** For all $x, y \in \mathcal{X}$, and any $u, v \in \mathcal{U}$, if $u(x) = v(x)$ and $u(y) = v(y)$, then $x \succeq_u y$ if and only if $x \succeq_v y$.
- (SP) **Strong Pareto:** For all $u \in \mathcal{U}$ and $x, y \in \mathcal{X}$, $u^s(x) \geq u^s(y)$ for all $s \in \mathcal{S}$ implies $x \succeq_u y$. If, in addition, at least one comparison is strict, then $x \succ_u y$.
- (A) **Anonymity:** For any permutation σ of $\mathcal{S}$, and any $u \in \mathcal{U}$, $F(u^1, \dots, u^S) = F(u^{\sigma(1)}, \dots, u^{\sigma(S)})$.

Our next result shows that, restricted to comparable profiles, an Arrovian social planner who satisfies (CUC) is necessarily utilitarian.²⁵

Theorem 2. *Suppose $F : \mathcal{C} \rightarrow \mathcal{P}$ is a social welfare functional for comparable profiles. Then F satisfies (CUC), (IIA), (SP), and (A) if and only if:*

$$x \succeq_u y \iff \sum_{s \in \mathcal{S}} u^s(x) \geq \sum_{s \in \mathcal{S}} u^s(y),$$

for all $x, y \in \mathcal{X}$ and $u \in \mathcal{C}$.

Methodologically, the proof of Theorem 2 follows closely that of d’Aspremont and Gevers (1977). The key difference is that, rather than proceeding from an unrestricted domain, we consider only comparable profiles of smooth utilities, and likewise require the resulting social preference to admit a smooth and critical-point-free representation.

Relative to the social choice literature, the novelty of Theorem 2 is in its use of our notion of comparability to justify the assumption of unit-comparability, rather

²⁵Note that both (CUC), via Proposition 2, and (A), remain well-defined restricted to $\mathcal{C}$.

than imposing it exogenously.²⁶ This provides a conceptual basis for interpreting the sum of these utilities as an ethically meaningful quantity.²⁷ However, our definition of comparability presents a wrinkle—not all profiles are comparable. Thus [Theorem 2](#) serves to verify that the classical characterization of utilitarianism remains valid, despite this restriction.

4.2 A Characterization of Comparability

Suppose the planner observes a profile $(u^1, \dots, u^S) \in \mathcal{U}$. At any $x \in \mathcal{X}$, let $r^S(x)$ denote the rank of the $d \times S$ matrix whose columns are the gradients of each u^s :

$$r^S(x) = \text{rank} \begin{bmatrix} \vdots & & \\ \cdots & \nabla u^s(x) & \cdots \\ \vdots & & \end{bmatrix}. \quad (7)$$

We say that the profile has **constant rank** if $r^S(x)$ is constant. This is a mild regularity condition which rules out singularities where the dimension of the space spanned by the normal vectors of the underlying indifference surfaces changes discretely. For any profile, the set of allocations at which $r^S(x)$ is locally constant is always open and dense in $\mathcal{X}$.²⁸

Theorem 3. *Suppose that $(u^1, \dots, u^S)$ has constant rank. Then the profile is interpersonally comparable if, and only if, for all $x \in \mathcal{X}$:*

$$0 \notin \text{Aff} \left\{ \nabla u^1(x), \dots, \nabla u^S(x) \right\}. \quad (8)$$

[Theorem 3](#) provides a simple, geometric characterization of when a constant-rank profile is interpersonally comparable: the affine hull of the gradients of each utility must never contain the zero vector. This serves as a straightforward test for verifying the comparability of a given profile.

²⁶See, e.g., discussion in [Fleurbaey and Hammond \(2004\)](#).

²⁷A long-standing critique of utilitarianism is that, absent some notion of utility unit, it is unclear why any particular tuple of representations should be singled out for aggregation. Arrow writes: “I’d like to be utilitarian but the only problem is I have nowhere those utilities come from. . . . What are those objects we are adding up? I have no objection to adding them up if there’s something to add.” ([Kelly and Arrow 1987](#), p. 59). Our notion of comparability provides a theoretical basis for ensuring such sums are meaningful.

²⁸See, e.g., [Lee \(2012\)](#) Theorem 4.14.

By the Fredholm alternative (e.g. [Gale 1989](#), Theorem 2.5), (8) holds precisely when, at each allocation $x \in \mathcal{X}$, there exists some vector whose dot product with every gradient is equal to one. Such a vector may be regarded as a choice of common, ‘infinitesimal numeraire’ for the profile at x . By smoothly selecting such a vector at each allocation, we obtain a vector field which is the generator $\dot{\varphi}$ of some common utility unit, φ .

In [Appendix G](#), we provide a complementary description of comparable profiles. We show a profile $(u^1, \dots, u^S)$ is comparable, with common unit φ , if and only if there exists a global change of coordinates on $\mathcal{X}$ which (i) simultaneously renders each u^s quasilinear, with respect to a common numeraire good, and (ii) maps each transfer function φ_t into a translation along this numeraire’s axis. Thus our notion of comparability may conceptually be regarded as simply requiring quasilinearity, but only up to change of coordinates.

5 Existence of Common Numeraires

Suppose a planner observes only a preference profile $(\succsim^1, \dots, \succsim^S)$. For any choice of common virtual numeraire, [Theorem 1](#) guarantees the existence of a unique profile of comparable representations, up to individual additive constants. By [Theorem 2](#), these uniquely identify the social preference chosen by an Arrovian planner.²⁹ Thus when multiple common numeraires exist, there may be multiple, equally justifiable utilitarian social preferences.

We consider a social planner who is selecting among small policy shifts from a status quo allocation. Such choices may be viewed as determining the *direction* of reform.³⁰ We will seek to determine whether any, and if so, how many, profiles of comparable representations exist, locally about the status quo, for a given preference profile. This, in turn, characterizes the range of valid utilitarian objectives such a planner could entertain, when faced with only ordinal information.

²⁹In particular, the additive constants have no effect on the resulting social preference.

³⁰Such problems of local welfare analysis have a long history in economics, including in welfare economics, cost-benefit analysis, and optimal taxation; see, for example, [Guesnerie \(1977\)](#); [Milleron \(1970\)](#); [Arrow and Lind \(1970\)](#); [Radner \(1993\)](#). Recent contributions include [Saez and Stantcheva \(2016\)](#), [Schlee and Khan \(2023\)](#), and [Sher \(2024\)](#).

For ease of exposition, we illustrate our results in the special case of profiles which admit a particular kind of locally separable representation. More generally, in [Appendix H](#), we provide a complete description of the set of common, local virtual numeraire, for general profiles. This fully characterizes all possible utilitarian objectives for problems of local welfare analysis.

Suppose, then, that x_0 is some status quo allocation, and there exists a subprofile of preferences $(\succsim^1, \dots, \succsim^{d+1})$ such that, for some (hence any) choice of utilities, the gradients of the first d representations are linearly independent.³¹ We say such a subprofile admits a **locally separable representation** if there exists some neighborhood $\mathcal{N}$ of x_0 , utility representations $v^1, \dots, v^d$ and v^{d+1} , of $\succsim^1|_{\mathcal{N}}, \dots, \succsim^{d+1}|_{\mathcal{N}}$, and vector $\lambda \in \{-1, 1\}^d$, such that:

$$v^{d+1}(x) = \sum_{i=1}^d \lambda^i v^i(x), \quad (9)$$

for all $x \in \mathcal{N}$. In such cases, we may view the locally defined utility v^{d+1} as an additively separable function, but of the first d utilities, rather than the original coordinates on $\mathcal{X}$.

When $d = 2$, such a subprofile admits a separable representation precisely when the three families of indifference curves, corresponding to each of the three preferences, satisfy the hexagon condition of [Debreu \(1959\)](#) on some neighborhood of x_0 . When $d \geq 3$, the existence of a locally separable representation is characterized by a simple modification of Theorem 1 of [Goldman and Uzawa \(1964\)](#).

Our next result characterizes the set of common, local numeraire for any such subprofile. Since any common numeraire for the full profile must also remain so for any such subprofile, this ensures all possible common numeraire belong to a simple, parametric family.

Theorem 4. *Suppose that $(\succsim^1, \dots, \succsim^{d+1})$ admits a locally separable representation (9) on some connected neighborhood $\mathcal{N}$ of x_0 on which $v(x) = (v^1(x), \dots, v^d(x))$ is smoothly invertible. Then a virtual commodity φ on $\mathcal{N}$ is a common virtual numeraire*

³¹By possibly relabeling our original profile, we may suppose this subprofile consists of the first $d + 1$ preferences.

if and only if:

$$\varphi_t(x) = v^{-1}\left(e^{at}v(x) + \frac{e^{at}-1}{a}b\right) \quad \text{and} \quad \begin{cases} av(x) + b \gg 0 \\ \lambda \cdot (av(x) + b) > 0, \end{cases}$$

for scalars $a, b^1, \dots, b^d$ and all $x \in \mathcal{N}$.³²

Theorem 4 shows that, when some subprofile admits a locally separable representation, every common, local virtual numeraire is constrained to take on a simple, parametric form. More generally, when (9) holds on $\mathcal{N} = \mathcal{X}$, it fully characterizes the subprofile's common numeraires, providing a parametric representation for any common numeraire of the full profile. It also provides a simple description, for any valid choice of parameters $(a, b^1, \dots, b^d)$, of the comparable profile denominated in that choice of unit.

Corollary 1. *For parameters $(a, b^1, \dots, b^d)$ defining a common numeraire, the unique, normalized, comparable representations with this unit are given by:*

$$u^i(x) = \begin{cases} \frac{1}{a} \ln \left(\frac{a v^i(x) + b^i}{a v^i(x_0) + b^i} \right) & a \neq 0 \\ \frac{v^i(x) - v^i(x_0)}{b^i} & a = 0 \end{cases},$$

for all $i = 1, \dots, d+1$, where $b^{d+1} \equiv \lambda \cdot b$.

For many classes of preferences, existence of a subprofile admitting such a representation is simply a matter of society being large enough. In such cases, **Theorem 4** describes the common numeraires of any sufficiently diverse profile. Examples include Cobb-Douglas preferences over consumption bundles, and expected utility preferences over lotteries.

Example 4. Suppose $\mathcal{X} = \mathbb{R}_{++}^2$, $x_0 = (1, 1)$, and let $(\succsim^1, \succsim^2, \succsim^3)$ be represented by Cobb-Douglas utilities:

$$\tilde{u}^i(x^1, x^2) = (x^1)^{\alpha^i} (x^2)^{1-\alpha^i} \quad \text{and} \quad \tilde{u}^3(x^1, x^2) = (x^1)^\beta (x^2)^{1-\beta},$$

for $i = 1, 2$, where we assume that $0 < \alpha^1 < \beta < \alpha^2 < 1$. By direct calculation:

$$\ln \tilde{u}^3 = \gamma^1 \ln \tilde{u}^1 + \gamma^2 \ln \tilde{u}^2,$$

³²When $a = 0$, we define φ_t to be $v^{-1}(v(x) + bt)$.

where $\gamma^1 = \frac{\alpha^2 - \beta}{\alpha^2 - \alpha^1}$ and $\gamma^2 = \frac{\beta - \alpha^1}{\alpha^2 - \alpha^1}$. Thus we may take $v^i = \gamma^i \ln \tilde{u}^i$, for $i = 1, 2$, and $v^3 = v^1 + v^2$. By [Theorem 4](#), every common, local virtual numeraire φ takes the form:

$$\varphi_t(x) = v^{-1} \left(e^{at} v(x) + \frac{e^{at} - 1}{a} b \right),$$

subject to the positivity constraint $a\gamma^i \ln \tilde{u}^i + b^i > 0$ for all $i = 1, 2$.³³ Since we have a separable representation globally, [Theorem 4](#) characterizes globally-defined numeraires. Now, because $\ln \tilde{u}^i$ takes on every real value, this positivity constraint forces $a = 0$. Thus, every globally-defined common numeraire takes the form:

$$\varphi_t(x) = \begin{bmatrix} e^{c^1 t} x^1 \\ e^{c^2 t} x^2 \end{bmatrix},$$

where c^1 and c^2 depend on the choice of $b \gg 0$.³⁴ By [Corollary 1](#), the normalized, comparable utilities associated with any such choice of parameters are:

$$u^i = \frac{\gamma^i}{b^i} \ln \tilde{u}^i, \quad i = 1, 2, \quad \text{and} \quad u^3 = \frac{\gamma^1 \ln \tilde{u}^1 + \gamma^2 \ln \tilde{u}^2}{b^1 + b^2}.$$

Thus the planner should not use the sum of the original Cobb-Douglas utilities $\tilde{u}^s$ as an objective, as these fail to be comparable. Instead, they should maximize the sum of the comparable representations u^s or, equivalently, maximize the Nash product of the Cobb-Douglas utilities, with weights reflecting the particular choice of numeraire, as determined by the parameters b . ■

5.1 Non-Separable Profiles

The existence of a separable profile ensures the problem of finding a common, local numeraire takes on a rigid, linear form. Given a representation (9), by considering the local utility possibility space $v(\mathcal{N})$, rather than $\mathcal{N}$ itself, the indifference curves of all $d + 1$ preferences become hyperplanes. In $(v^1, \dots, v^d)$ -coordinates, [Theorem 4](#) simply says a virtual commodity is a common numeraire precisely when its generator has the form:

$$\dot{\varphi}(v) = av + b.$$

³³Here, since $\alpha^1 < \beta < \alpha^2$, $\lambda = (1, 1)$ hence the final constraint $\lambda \cdot (av(x) + b) > 0$ is redundant.

³⁴Concretely:

$$c^1 = \frac{(1 - \alpha^1)b^2/\gamma^2 - (1 - \alpha^2)b^1/\gamma^1}{\alpha^2 - \alpha^1} \quad \text{and} \quad c^2 = \frac{\alpha^2 b^1/\gamma^1 - \alpha^1 b^2/\gamma^2}{\alpha^2 - \alpha^1}.$$

These may be obtained by explicitly computing the inverse v^{-1} .

Conversely, when no subprofile admits a locally separable representation, determining the existence of common numeraire is governed by an overdetermined system of partial differential equations. In general, no solutions to these equations exist unless the system satisfies particular integrability, or compatibility conditions. Using geometric techniques, we characterize the solution set of such systems in [Appendix H](#).

This provides criteria not only for determining the set of local numeraires for a given preference profile, but also for obtaining necessary and sufficient conditions on *families* of profiles, for the existence of some tuple of comparable local representations. For example, when $\mathcal{X} = \mathbb{R}_{++}^3$, we show that a profile of preferences represented by utilities of the form:

$$u^i(x) = x^i, \quad i = 1, 2, 3, \quad \text{and} \quad u^4(x) = f(x^1 + x^2) + x^3,$$

for some smooth, increasing and strictly concave f , admits a common numeraire if, and only if, f is additionally the Bernoulli utility of a HARA expected utility function ([Merton 1971](#)).³⁵

6 Examples & Applications

6.1 Aggregation of Expected Utility Preferences

[Theorem 2](#) shows that, when presented with a comparable profile of utilities, an Arrovian planner will aggregate them linearly. However, as noted, when only a profile of ordinal preferences is observed, there may be multiple, distinct, comparable profiles of representations, each corresponding to a different utility unit. This can lead to multiple, equally valid choices of objective for such a planner.³⁶

One notable example is the case of expected utility preferences. Indeed, many of the most common, and well-known approaches to aggregating such preferences (e.g. [Harsanyi 1953, 1955](#); [Kaneko and Nakamura 1979](#); [Dhillon and Mertens 1999](#)) all arise as utilitarian aggregation of comparable profiles, differing only in their implicit choice of numeraire unit.

³⁵Moreover, when non-empty, the set of local numeraires always depends smoothly on exactly three parameters. We prove this in [Appendix H](#).

³⁶See [Sen \(1970\)](#) for a discussion.

Let $\mathcal{Z} \subset \mathbb{R}$ denote a finite set of monetary prizes, and suppose that $\mathcal{X}$ is the interior of the simplex $\Delta(\mathcal{Z})$. Let $(\succsim^1, \dots, \succsim^S)$ be a profile of monotone expected utility preferences on $\mathcal{X}$. We will denote the highest prize in $\mathcal{Z}$ by z^* and the lowest by z_* . Finally, let $(\tilde{u}^1, \dots, \tilde{u}^S)$ denote any profile of expected utility representations.

Common virtual numeraire for such profiles can take several forms. Suppose $\gamma \in \mathbb{R}^{\mathcal{Z}}$ is any zero-sum vector such that $p + \gamma \succ^s p$ for some $p \in \mathcal{X}$ and every $s \in \mathcal{S}$. Then any **translation** commodity, $\varphi_t(p) = p + t\gamma$, defines a common numeraire.³⁷ The unique, comparable utilities denominated in this unit are:

$$u^s(p) = \frac{1}{\gamma \cdot \nabla \tilde{u}^s} \tilde{u}^s(p),$$

up to individual additive constants.

In this case, [Theorem 2](#) calls for the planner to maximize a weighted sum of the expected utilities $\tilde{u}^s$, as in [Harsanyi \(1953, 1955\)](#). Our results clarify that the indeterminacy of the weights is a consequence of the fact that, in general, there are many possible, valid choices of translation direction γ , and the social weights $1/(\nabla \tilde{u}^s \cdot \gamma)$ depend upon the particular selection.³⁸

In the special case when γ is proportional to $\delta_{z^*} - \delta_{z_*}$, the comparable profiles are still of expected utility form. However, now the range of each u^s is an open interval of common length.³⁹ As a consequence, the utilitarian sum of comparable utilities is, up to an affine transform, equal to the objective of a *relative* utilitarian ([Dhillon and Mertens 1999](#)).

Alternatively, defining $\varphi_t(p) = e^t p + (1 - e^t) \delta_{z_*}$ also yields a common numeraire. Here, φ_t corresponds to a **dilation** about the least-desirable prize, δ_{z_*} .⁴⁰ As a conse-

³⁷Here, for any lottery p , the interval $\mathcal{D}^{(p)}$ is simply:

$$\left(\max_{\{i: \gamma_i > 0\}} \left(\frac{-p_i}{\gamma_i} \right), \min_{\{i: \gamma_i < 0\}} \left(\frac{-p_i}{\gamma_i} \right) \right).$$

Since γ is zero-sum and not identically equal to zero, the max and min are taken over non-empty sets.

³⁸See also [Mononen \(2025\)](#) for more discussion.

³⁹Concretely, if $\varphi_t(p) = p + t\varepsilon(\delta_{z^*} - \delta_{z_*})$, for some $0 < \varepsilon < 1$, then:

$$t = u^s(\varphi_t(p)) - u^s(p) = u^s(p + t\varepsilon(\delta_{z^*} - \delta_{z_*}) - p) = t\varepsilon[u^s(\delta_{z^*}) - u^s(\delta_{z_*})],$$

hence the range of each u^s is an open interval of length $1/\varepsilon$, for all $s \in \mathcal{S}$.

⁴⁰The associated commodity domain is defined by $\mathcal{D}^{(p)} = (-\infty, -\ln(1 - p_{z^*}))$.

quence, the comparable profiles are no longer of expected utility form. Instead, they are given by $u^s(p) = \ln(\tilde{u}^s(p) - \tilde{u}^s(\delta_{z_*}))$, up to individual additive constants. In this case, [Theorem 2](#) then calls for maximizing the Nash social welfare function ([Kaneko and Nakamura 1979](#)).

Thus, despite these methods of aggregation admitting distinct, independent axiomatizations, we find each arises as a special case of utilitarianism of comparable profiles. In light of [Theorem 2](#), this provides a novel, transparent, and unifying axiomatic description for each of these theories. In particular, our results clarify, and justify, the precise role that interpersonal comparisons play in each.⁴¹

A frequent critique of expected utilities in social choice is that this selection of representation is arbitrary, without ordinal justification ([Arrow 1951](#); [Sen 1976](#); [Roemer 2008](#); [Weymark 2005](#)). One implication of our results is that (normalized) expected utilities, and their logarithms, occupy a privileged position as among the only interpersonally comparable profiles, whereas more general monotone transforms are not.

In light of this, a natural question is whether there exist other methods of aggregation, corresponding to non-translationary, or dilationary, numeraire.⁴² For any sufficiently rich profile, we now show that these two types of numeraire, and their combinations, are the only possibilities.

Suppose there exists a locally separable subprofile, $(\succsim^1, \dots, \succsim^{d+1})$. Any such representation (9) holds globally, where each v^i , $i = 1, \dots, d+1$ is an expected utility function. Without loss of generality, we may normalize each so that $v^i(\delta_{z_*}) = 0$. Then by [Theorem 4](#), every common virtual numeraire for this subprofile is of the form:

$$\varphi_t(p) = v^{-1}\left(e^{at}v(p) + \frac{e^{at} - 1}{a}b\right),$$

where $av(p) + b \gg 0$, for the parameters $a, b^1, \dots, b^d$. When $a = 0$, every such numeraire is a translation:

$$\varphi_t(p) = v^{-1}(v(p) + tb) = p + t\gamma,$$

⁴¹This has been the subject of extensive debate, e.g. [Harsanyi \(1975\)](#); [Sen \(1976, 1977\)](#); [Roemer \(2008\)](#); [Weymark \(2005\)](#).

⁴²Equivalently, do there exist other profiles of comparable representations, beyond normalized expected utilities and their logarithms, which can be summed to obtain new forms of social preference.

as v is affine. In this case, [Corollary 1](#) confirms that the comparable representations are indeed of expected utility form. If instead $b = 0$, then φ_t is a dilation about δ_{z_*} , as:

$$\varphi_t(p) = v^{-1}(e^{at}v(p) + (1 - e^{at})v(\delta_{z_*})) = e^{at}p + (1 - e^{at})\delta_{z_*},$$

as each v^i is affine, and $v^i(\delta_{z_*}) = 0$. Here, as noted above, the comparable profiles are the normalized logarithms of the expected utilities v^i . Thus the only other common utility units are precisely those φ that are of ‘mixed’ form, consisting of both a dilation and translation.

In particular, any general, interpersonally-valid method of aggregating expected utility preferences must rely on comparisons denominated in a common unit of the above form. The sum of expected utilities, and the Nash social welfare function, correspond to the two extremal cases (either a or b equal to zero) within this class. The full parametric family of possible approaches is then obtained simply by summing utilities given in [Corollary 1](#).

6.2 Tax Reform

A large literature studies the direction of marginal tax reform under revenue, feasibility, and distributional constraints ([Guesnerie 1977](#); [Tirole and Guesnerie 1981](#); [King 1983](#); [Ahmad and Stern 1984](#)). More recent work provides necessary and sufficient conditions for the existence of Pareto-improving tax reforms ([Bierbrauer et al. 2023](#)) and studies the properties of generalized welfare weights incorporating non-utilitarian criteria ([Saez and Stantcheva 2016](#); [Sher 2024](#)).

Our theory of comparability supplies an alternative foundation for the household welfare gradients entering a tax-reform problem. Rather than suppose that money is itself an interpersonally valid unit, or introduce household-specific welfare weights on the basis of external considerations, we suppose households’ direct utilities possess a common measurement unit $(\mathcal{D}, \varphi)$. This allows us to meaningfully compare households’ marginal utilities of income and, given this common unit, [Theorem 2](#) justifies aggregating utility changes linearly.

Let $\mathcal{X} = \mathbb{R}_{++}^d$. A market state is a tuple $(p^1, \dots, p^d, w^1, \dots, w^S) \gg 0$, which we denote by (p, w) . Suppose that each household’s direct utility $u^s : \mathcal{X} \rightarrow \mathbb{R}$ is comparable, with common utility unit $(\mathcal{D}, \varphi)$. We will assume that at every market

state, each household's Marshallian demand $x^s(p, w^s)$ belongs to $\mathcal{X}$, i.e. is strictly interior. Finally, let $v^s(p, w^s) = u^s(x^s(p, w^s))$ denote the indirect utilities associated with the comparable utilities u^s .

For simplicity, we consider proportional price reforms. Given some status quo market state (p_0, w_0) , a vector $a \in \mathbb{R}^d$ defines a price path via:

$$p^i(t) = p_0^i e^{ta^i}, \quad i = 1, \dots, d,$$

thus a positive a^i corresponds to a proportional increase in the price of the i th commodity. Define the **price-exposure** of household s as the vector:

$$\mu^s = \frac{\text{diag}(p_0) x_0^s}{\dot{\varphi}(x_0^s) \cdot p_0}, \quad (10)$$

where $x_0^s \equiv x^s(p_0, w_0^s)$. Thus, for any household, the first-order effect of a price reform a is given by:

$$h^s(a) = \left. \frac{\partial}{\partial t} v^s(p(t), w_0^s) \right|_{t=0} = -\mu^s \cdot a.$$

These quantities admit meaningful interpersonal comparisons. To see this, by differentiating both sides of (3) at x_0^s with respect to t , and evaluating at $t = 0$, we obtain $\dot{\varphi}(x_0^s) \cdot \nabla u^s(x_0^s) = 1$, and hence, by plugging in the first-order condition from the household's utility maximization problem at (p_0, w_0) :

$$\lambda^s(p_0, w_0^s) = \frac{1}{\dot{\varphi}(x_0^s) \cdot p_0},$$

where by the envelope theorem, λ^s denotes the marginal utility of wealth. Thus the numerator in (10) is simply the vector of household expenditures at the status quo, and the denominator is the dollar cost, at prices p_0 , of an infinitesimal unit of the common numeraire. Thus the vector μ^s reflects expenditures denominated in the common utility unit φ .

This can yield new insights relative to standard means of computing the impact of a reform. For example, even in simple circumstances, a reform selected on the basis of common-unit utilitarianism may differ significantly from, e.g., the reform that maximizes the sum of money metrics.⁴³

⁴³This is conceptually distinct from, e.g., equity concerns arising from the general non-concavity of money metric sums; see [Schlee and Khan \(2023\)](#) and references therein.

Example 5. Suppose $\mathcal{X} = \mathbb{R}_{++}^2$, and there are three households, with utilities:

$$\tilde{u}^s(x) = \left(\sqrt{\theta^s x^1} + \sqrt{(1 - \theta^s)x^2} \right)^2,$$

where $\theta = (.2, .5, .8)$. For any $p \gg 0$, the expenditure functions associated with these $\tilde{u}^s$ are given by:

$$e^s(p, \tilde{u}) = \frac{\tilde{u}}{\frac{\theta^s}{p^1} + \frac{1-\theta^s}{p^2}}.$$

Fix the reference price vector $\bar{p} = (2, 1)$. The corresponding money metric utilities, in these reference prices, are then:

$$m^s(x) = \left(\frac{2}{2 - \theta^s} \right) \tilde{u}^s(x).$$

By direct calculation, the m^s may be seen to be incomparable.⁴⁴ However, the normalization $u^s = \ln m^s$ are straightforwardly comparable, under the radial numeraire $\varphi_t(x) = e^t x$.

Suppose the status quo market state is given by $p_0 = (1, 1)$, and $w_0 = (1, 1, 2)$. At the status quo, each household's demand $x_0^s = \theta^s w_0^s, (1 - \theta^s)w_0^s$, and hence aggregate status quo demands are $Q = (2.3, 1.7)$. Consider two indirect policies which affect the market through proportional price changes:

$$a^A = \left(\frac{10}{23}, 0 \right) \quad \text{and} \quad a^B = \left(0, \frac{10}{17} \right).$$

Each reform raises one unit of revenue, up to first order.⁴⁵ Now, under φ ,

$$h^s(a) = -\theta^s a^1 - (1 - \theta^s)a^2,$$

thus the utilitarian welfare under φ of each policy is given by:

$$\sum_{s=1}^3 h^s(a^A) = -\frac{15}{23} \quad \text{and} \quad \sum_{s=1}^3 h^s(a^B) = -\frac{15}{17},$$

and hence the planner would select policy A . Conversely, at the status quo, the money metric utility of each household is $m^s(x_0^s) = \left(\frac{2}{2 - \theta^s} \right) w_0^s$, hence the marginal utility of

⁴⁴For example, at $\bar{x} = (1, 1)$, the gradients satisfy $6\nabla m^1(\bar{x}) - 9\nabla m^2(\bar{x}) + 4\nabla m^3(\bar{x}) = 0$, implying the zero vector belongs to their affine hull. Thus by [Theorem 3](#) they are not comparable.

⁴⁵That is, $Q \cdot a^A = 1 = Q \cdot a^B$.

wealth of each household is $(\frac{2}{2-\theta^s})$. Thus the first-order money metric welfare effect is given by:

$$h_{mon}^s(a) = -\left(\frac{2w_0^s}{2-\theta^s}\right)(\theta^s a^1 + (1-\theta^s)a^2)$$

Summing these across households yields a criterion which instead selects policy B .⁴⁶ However, the money metric sum which generated this conclusion is incoherent. There is no way to rationalize the welfare units h_{mon}^s as denominated in any common unit. This highlights a novel insight arising from our notion of comparability: even common-reference-price money metric utilities may fail to be comparable, leading to judgment reversals relative to the comparable-utilitarian benchmark. This shows that any characterization of the sum of money metric utilities (e.g. [Bosmans et al. 2018](#)) necessarily requires the introduction of external, normative considerations. ■

7 Conclusion

This paper develops a theory of objective measurement units for utility. We have shown that our theory is rich enough to provide an appropriate measurement unit for any choice of representation of any preference, and conversely that any virtual commodity which is, in our generalized sense, a numeraire for a preference is the utility unit for some unique (up to additive constant) choice of representation.

Given that virtual commodities can be defined objectively, without reference to any preference information, it is meaningful to ask when a given commodity is a utility scale for a profile of utilities. This gives rise to a natural notion of interpersonal comparability: a profile of utilities is comparable precisely when they can be rationalized as denominated in a common unit. In such cases, we show an Arrovian planner will always choose to maximize the sum of these utilities.

This both unifies and extends many existing results on aggregation and welfare. For example, we show that each of the most widely-studied methods of aggregating expected utility preferences arises as utilitarian aggregation of comparable profiles

⁴⁶Here,

$$\sum_{s=1}^3 h_{mon}^s(a^A) = -\frac{320}{207} \approx -1.55 \quad \text{versus} \quad \sum_{s=1}^3 h_{mon}^s(a^B) = -\frac{200}{153} \approx -1.31.$$

of utility representations. In particular, the differences in these approaches all arise purely due to different choices of implicit utility unit. However, our results also hold for general profiles of smooth preferences, significantly extending these ideas beyond the context of risk.

Our results suggest numerous questions for further inquiry. For instance, our notion of common utility scales provides a natural framework for considering interdependent preferences, where subjects care not only about their direct allocation, but also those of others. This suggests a pathway to studying *ordinal* preference profiles which admit representations reflecting phenomena such as altruism, envy, or inequality aversion.

It also provides means of determining which aggregate, ordinal welfare criteria admit meaningful cardinal extensions. For example, while envy-freeness is a black-and-white ordinal property, comparability allows for shades of gray, quantifying the *magnitude* of aggregate envy induced by an allocation. Such ideas may also be of use in developing general inefficiency-bounding theorems, such as those in [Gonczarowski and Segev \(2024\)](#). We leave the development of these and other ideas to future research.

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A A Flow Lemma

Virtual commodities, as defined in [Section 3.1](#), simply correspond to maximal flows on $\mathcal{X}$ (see Chapter 9 of [Lee 2012](#) for formal definitions). This gives rise to a one-to-one correspondence between virtual commodities $(\mathcal{D}, \varphi)$, and their generators, $\dot{\varphi}$.

Lemma 1. *Every virtual commodity admits a unique generator, and conversely, any vector field on $\mathcal{X}$ is the generator of a unique virtual commodity. Moreover, every virtual commodity $(\mathcal{D}, \varphi)$ satisfies:*

(i) *For any $x \in \mathcal{X}$, if $t \in \mathcal{D}^{(x)}$, then:*

$$\mathcal{D}^{(\varphi_t(x))} = \mathcal{D}^{(x)} - t = \{t' - t : t' \in \mathcal{D}^{(x)}\}.$$

(ii) *For all $t \in \mathbb{R}$, the set $\mathcal{X}_t = \{x \in \mathcal{X} : (x, t) \in \mathcal{D}\}$ is open in $\mathcal{X}$, and the map φ_t is a diffeomorphism $\mathcal{X}_t \rightarrow \mathcal{X}_{-t}$, with inverse φ_{-t} .*

Proof. See [Lee \(2012\)](#), Theorem 9.12. □

B Proof of [Theorem 1](#)

Proof. We first prove claim (i). Suppose $\succsim$ is a preference, and $(\mathcal{D}, \varphi)$ is a virtual numeraire for it. We will show there exists some representation v of $\succsim$ which satisfies [\(3\)](#). Since $\succsim$ is a preference, it admits some smooth, critical-point-free utility representation $u : \mathcal{X} \rightarrow \mathbb{R}$.

Suppose $x \sim y$. By (N.1), for any $t \in \mathcal{D}^{(x)} \cap \mathcal{D}^{(y)}$, we have $u(\varphi_t(x)) = u(\varphi_t(y))$. Differentiating both sides and evaluating at $t = 0$ yields $\dot{\varphi}(x) \cdot \nabla u(x) = \dot{\varphi}(y) \cdot \nabla u(y) > 0$.⁴⁷ Since x and y were arbitrary points belonging to a general level set, there exists some smooth function $h : \text{range}(u) \rightarrow \mathbb{R}_{++}$ such that $\dot{\varphi}(x) \cdot \nabla u(x) = (h \circ u)(x)$. Fix any $\bar{q} \in \text{range}(u)$. Then, defining $H : \text{range}(u) \rightarrow \mathbb{R}$ via:

$$H(q) = \int_{\bar{q}}^q \frac{1}{h(\xi)} d\xi$$

yields a strictly increasing, critical-point-free map. But by the chain rule:

$$\dot{\varphi}(x) \cdot \nabla (H \circ u)(x) = H'(u(x)) \dot{\varphi}(x) \cdot \nabla u(x) = \frac{1}{(h \circ u)(x)} \dot{\varphi}(x) \cdot \nabla u(x) = 1.$$

Thus $v = (H \circ u)$ represents $\succsim$ and has constant X -directional derivative equal to unity. Therefore:

$$v(\varphi_t(x)) = v(x) + \int_0^t (\dot{\varphi} \cdot \nabla v)|_{\varphi_\xi(x)} d\xi = v(x) + \int_0^t 1 d\xi = v(x) + t,$$

and hence v satisfies (3) as desired.

We now prove (ii). Toward this end, let u be an arbitrary utility representation of $\succsim$. It suffices to exhibit a virtual commodity $(\mathcal{D}, \varphi)$ such that u satisfies (3). Consider the vector field $F(x) = \nabla u(x) / \|\nabla u(x)\|_2^2$. Let φ denote the unique, smooth, maximal flow associated with the vector field F , with domain $\mathcal{D}$. By Theorem 9.12 of Lee (2012), this defines a virtual commodity. But now, by the fundamental theorem of calculus for line integrals:

$$u(\varphi_t(x)) - u(x) = \int_0^t (F \cdot \nabla u)|_{\varphi_\xi(x)} d\xi = \int_0^t 1 d\xi = t.$$

Thus $(\mathcal{D}, \varphi)$ and u satisfy (3) and hence $(\mathcal{D}, \varphi)$ is a virtual numeraire for $\succsim$.

Finally, suppose u and v are representations of $\succsim$, and both satisfy (3) for some common virtual numeraire $(\mathcal{D}, \varphi)$. Then $\dot{\varphi} \cdot \nabla u = 1 = \dot{\varphi} \cdot \nabla v$ pointwise. By the chain rule, ∇u and ∇v are additionally everywhere collinear, implying they must be equal. As a consequence, u and v are equal up to a constant of integration. $\square$

⁴⁷Positivity follows by noting that if, e.g., $\dot{\varphi}(x) \cdot \nabla u(x) = 0$, then (N.1) would imply the same of all x' on the same indifference surface near x . Thus, for any such x' , $\dot{\varphi}(x')$ is tangent to the x -indifference surface, thus the trajectory of $\dot{\varphi}$ starting from x (i.e. $\varphi_t(x)$ for small t) must belong to this indifference surface, violating (N.2).

C Proof of Theorem 2

We will rely on the following technical lemma.

Lemma 2. *For any distinct $x_1, \dots, x_K \in \mathcal{X}$, and any $A \in \mathbb{R}^{K \times S}$, there exists a profile $u \in \mathcal{C}$ such that:*

$$a_{ks} = u^s(x_k)$$

for all $s = 1, \dots, S$ and $k = 1, \dots, K$.

Proof. Let $V = \text{span}\{x - x' : x, x' \in \mathcal{X}\}$, and fix $\{x_1, \dots, x_K\} \subset \mathcal{X}$. Since $\dim V = \dim \mathcal{X} = d > 1$, there exists a vector $v \in V$ such that v is not parallel to $x_k - x_{k'}$, for all distinct $k, k' = 1, \dots, K$, and let $e_1, \dots, e_{d-1}, v$ denote any extension of v into a basis for V .

Now, let $x_0 \in \mathcal{X}$ be an arbitrary, fixed allocation. Then every $x \in \mathcal{X}$ can be written uniquely as:

$$x = x_0 + \sum_{i=1}^{d-1} y^i e_i + tv,$$

thus we may express each x uniquely in (y, t) coordinates. Writing $x_k = (y_k, t_k) = (y_k^1, \dots, y_k^{d-1}, t_k)$, note that for all $k \neq k'$, we have $y_k \neq y_{k'}$, as if this were not true, $x_k - x_{k'} = (t_k - t_{k'})v$, a contradiction.

Define, for each $k = 1, \dots, K$:

$$\lambda_k(y) = \prod_{\substack{k'=1 \\ k' \neq k}}^K \frac{\|y - y_{k'}\|^2}{\|y_k - y_{k'}\|^2}.$$

By the preceding argument, this is well-defined as its denominator is always non-zero. Moreover, for any $j, k = 1, \dots, K$, $\lambda_k(y_j)$ is one if $j = k$ and zero otherwise. Finally, let:

$$u^s(y, t) = t + \sum_{k=1}^K (a_{ks} - t_k) \lambda_k(y).$$

Then at $x_{k'}$, we have:

$$\begin{aligned} u^s(y_{k'}, t_{k'}) &= t_{k'} + \sum_{k=1}^K (a_{ks} - t_k) \lambda_k(y_{k'}) \\ &= t_{k'} + (a_{k's} - t_{k'}) \\ &= a_{k's} \end{aligned}$$

as desired. As each u^s is clearly t quasilinear, $\varphi_t(x) = x + tv$ is a common utility unit, and thus $u = (u^1, \dots, u^S) \in \mathcal{C}$. $\square$

C.1 Proof of Theorem 2

Proof. In light of Lemma 2, our proof essentially mirrors that of Theorem 3 of d'Aspremont and Gevers (1977). We describe the changes that are needed below.

Lemma 2 of d'Aspremont and Gevers (1977) considers only profiles whose utility values are specified at three alternatives. By Lemma 2 such profiles always exist in $\mathcal{C}$ hence their Lemma 2, in its original proof, goes through essentially unchanged. Thus our F satisfies their axiom (XNE).

Now, for any $a, b \in \mathbb{R}^S$ define $a R^* b$ if and only if there exist $x, x' \in \mathcal{X}$ and $u \in \mathcal{C}$ where $u(x) = a$, $u(x') = b$, and $x \succeq_u x'$. By Lemma 2 such x, x' and u can always be found. Then Lemma 3 of d'Aspremont and Gevers (1977) goes through unchanged for this relation R^* .

The proof of Lemma 4 in d'Aspremont and Gevers (1977) is the sole other point in which a slightly modified construction is needed. Writing I^* as the symmetric component of R^* , we wish to show that for any permutation $\sigma : \mathcal{S} \rightarrow \mathcal{S}$, and any $a \in \mathbb{R}^S$, $a I^* \sigma a$, where $\sigma a = (a^{\sigma(1)}, \dots, a^{\sigma(S)})$. It suffices to consider σ a transposition, as every permutation is a product of transpositions. Thus, let $b = \sigma a$ and by Lemma 2, suppose $u \in \mathcal{C}$ and $x, x' \in \mathcal{X}$ satisfies $u(x) = a$ and $u(x') = b$. Let $\sigma u = (u^{\sigma(1)}, \dots, u^{\sigma(S)})$. By (A), $\succeq_u = \succeq_{\sigma u}$. Since $\sigma^2 = \text{id}$, we have $\sigma u(x) = b$ and $\sigma u(x') = a$, hence:

$$a R^* b \iff x \succeq_u x' \iff x \succeq_{\sigma u} x' \iff b R^* a.$$

The claim for arbitrary permutations σ then follows. Lemma 5 of d'Aspremont and Gevers (1977) then goes through unchanged, and likewise the remainder of the proof of their Theorem 3. $\square$

D Proof of Theorem 3

Proof. First, note that if at some $x \in \mathcal{X}$, $0 \notin \text{Aff}\{\nabla u^1(x), \dots, \nabla u^S(x)\}$, then:

$$\sum_{s \in \mathcal{S}} c^s \nabla u^s(x) = 0 \implies \sum_{s \in \mathcal{S}} c^s = 0, \quad (11)$$

as otherwise the vector c could be rescaled to sum to unity. Conversely, if $0 \in \text{Aff}\{\nabla u^1(x), \dots, \nabla u^S(x)\}$, there exists $c \in \mathbb{R}^S$, such that:

$$\sum_{s \in \mathcal{S}} c^s \nabla u^s(x) = 0 \quad \text{and} \quad \sum_{s \in \mathcal{S}} c^s = 1.$$

Thus (11) is equivalent to (8).

Now, let $r = r(\mathcal{S})$. As the profile has constant rank, at every $x \in \mathcal{X}$, there exist $\nabla u^{s_1}, \dots, \nabla u^{s_r}$ which are linearly independent on some neighborhood of $x \in \mathcal{X}$. Thus, for every $x \in \mathcal{X}$, there exists an open, connected neighborhood U_x of x , and selection of r gradients $\nabla u^{s_1}, \dots, \nabla u^{s_r}$, such that these gradients remain linearly independent on U_x . For each $x \in \mathcal{X}$, let $\mathcal{A}_x$ denote the $d \times r$ matrix-valued function whose columns consist of these gradients.

Then, for every $(U_x, \mathcal{A}_x)$, and every $x' \in U_x$, the $r \times r$ matrix $\mathcal{A}_x^\top(x')\mathcal{A}_x(x')$ is invertible, hence we may define a function F_x on U_x via:

$$F_x(x') = \mathcal{A}_x(x') [\mathcal{A}_x^\top(x')\mathcal{A}_x(x')]^{-1} \mathbb{1},$$

where $\mathbb{1}$ denotes the column vector of r ones. This function is smooth and, by construction, $\mathcal{A}_x(x')^\top F_x(x') = \mathbb{1}$. Thus $F_x(x')$ has dot product one with every gradient $\nabla u^{s_1}(x'), \dots, \nabla u^{s_r}(x')$, at every $x' \in U_x$. Now, on U_x , every remaining gradient can be written as:

$$\nabla u^s(x') = \sum_{i=1}^r c^i(x') \nabla u^{s_i}(x'),$$

where the coefficient functions c^i are smooth. By (11), the weight functions must sum to unity, as $\nabla u^s(x') - \sum_{i=1}^r c^i(x') \nabla u^{s_i}(x') = 0$. Thus $F_x(x') \cdot \nabla u^s(x') = \sum_{i=1}^r c^i(x') F_x(x') \cdot \nabla u^{s_i}(x') = \sum_{i=1}^r c^i(x') = 1$. Thus, each F_x has dot product equal to one with every gradient $\nabla u^1, \dots, \nabla u^S$ at every point of U_x .

Since $\mathcal{X}$ is a subset of Euclidean space it is paracompact, therefore the open cover $\mathcal{U} = \{U_x\}_{x \in \mathcal{X}}$ admits a locally finite refinement, $\tilde{\mathcal{U}} = \{\tilde{U}_i\}_{i \in I}$. For each $i \in I$, select any x_i such that $\tilde{U}_i \subseteq U_{x_i}$. Now, let $\{\psi_i\}_{i \in I}$ denote a smooth partition of unity subordinate to $\tilde{\mathcal{U}}$. Then:

$$F(x) = \sum_{i \in I} \psi_i(x) F_{x_i}(x)$$

is smooth and well-defined, as the open cover $\{\tilde{U}_i\}_{i \in I}$ is locally finite, hence for any $x \in \mathcal{X}$, the sum is taken over only finitely many non-zero terms. Moreover, as at

every point it is a convex combination of vectors which have dot product equal to one with respect to every gradient, so too does F . Then, by an identical argument to that appearing in the proof of [Theorem 1](#), there is then a unique virtual commodity $(\mathcal{D}, \varphi)$ on $\mathcal{X}$ such that $\dot{\varphi} = F$ pointwise. This φ then necessarily satisfies (3) for every u^s .

Conversely, suppose there exists some common utility unit $(\mathcal{D}, \varphi)$ for every u^s . Thus, for every u^s , we have $\dot{\varphi} \cdot \nabla u^s = 1$. For sake of contradiction, suppose, now that $0 \in \text{Aff}\{\nabla u^1(x), \dots, \nabla u^S(x)\}$. By (11) there must be some $c \in \mathbb{R}^S$ such that $\sum_s c^s \nabla u^s(x) = 0$ and $\sum_s c^s = 1$. But this means $0 = \dot{\varphi} \cdot \sum_s c^s \nabla u^s = \sum_s c^s \dot{\varphi} \cdot \nabla u^s = \sum_s c^s = 1$, a contradiction. Thus $0 \notin \text{Aff}\{\nabla u^1(x), \dots, \nabla u^S(x)\}$, as desired. $\square$

E Proof of [Theorem 4](#)

Proof. Suppose $\mathcal{N} \subseteq \mathcal{X}$ is a connected neighborhood of x_0 on which (9) holds, for utilities $v^1, \dots, v^{d+1}$ and some vector $\lambda \in \{-1, 1\}^d$. Define $v(x) = (v^1(x), \dots, v^d(x))$, and let $\mathcal{V} = v(\mathcal{N})$. Let φ be a virtual commodity on $\mathcal{V}$ such that it is a common numeraire for the preferences $\succsim^1, \dots, \succsim^d$ represented by the coordinate functions v^i , $i = 1, \dots, d$ and $\succsim^{d+1}$ represented by $\lambda^1 v^1 + \dots + \lambda^d v^d$. By (N.1) applied to each $\succsim^i$, $i = 1, \dots, d$, we have:

$$\dot{\varphi}^i(v) = F^i(v), \quad (12)$$

for some tuple of functions $(F^1, \dots, F^d)$, defined on $\mathcal{V}$ and satisfying $F_j^i = 0$ for all $i \neq j$. But by (N.1) for $\succsim^{d+1}$, we obtain:

$$F^{d+1}(v) = \sum_{i=1}^d \lambda^i F^i(v^i)$$

is constant on subsets of vectors with constant λ dot product in $\mathcal{V}$. Let ξ^i denote the d -vector of all 0's except for its i th component, which is 1. Then, for all $i \neq j$:

$$DF^{d+1}(v)(\lambda^i \xi^i - \lambda^j \xi^j) = F_i^i(v) - F_j^j(v) = 0.$$

Since v is a diffeomorphism on $\mathcal{N}$, $\mathcal{V}$ is open. Thus each coordinate around any fixed $v(x)$ can be independently varied, hence the above implies every F_i^i is locally constant and $F_i^i = F_j^j$ for all i, j . Since $\mathcal{N}$ is connected, we obtain that each F^i is the restriction to $\mathcal{V}$ of a function:

$$F^i(v) = av^i + b^i.$$

By (N.2), for each $i = 1, \dots, d$, we have $av^i + b^i > 0$, as well as $\lambda \cdot (av^i + b) > 0$. Conversely, clearly any φ satisfying (12) also satisfies (N.1) and (N.2). Thus every virtual numeraire takes this form. The claim then follows from solving the ordinary differential equation (12) and conjugating back to $\mathcal{N}$ via v . $\square$

F Proposition Proofs

F.1 Proof of Proposition 1

Proof. Fix a choice of utility representation $u : \mathcal{X} \rightarrow \mathbb{R}$ for $\succsim$, and suppose $y \succsim x$ for some $x, y \in \mathcal{X}$. First, note if $x \sim y$, then the singleton sequence $(x_1, t_1) = (x, 0)$ trivially belongs to $\mathcal{D}$ and suffices for the claim. Thus suppose $y \succ x$, and consider the interval $A = [u(x), u(y)] \subset \mathbb{R}$. By convexity of $\mathcal{X}$, for every $a \in A$ there exists some $z_a \in \mathcal{X}$ with $u(z_a) = a$. Now, for each $a \in A$, let:

$$I_a = u(\varphi(z_a, \mathcal{D}^{(z_a)}))$$

for an arbitrary choice of $z_a \in u^{-1}(a)$. Since u is continuous, and by (N.2), $u(\varphi(z_a, t))$ is strictly increasing in t , the fact $\mathcal{D}^{(z_a)}$ is an open interval containing zero implies $I_a \subseteq \mathbb{R}$ is an open interval containing a .

Thus $\{I_a\}_{a \in A}$ defines an open cover of the compact interval A , and hence there exists a finite $A' \subset A$ such that $\{I_a\}_{a \in A'}$ forms a finite subcover; by the Lebesgue number lemma, there also exists a partition:

$$u(x) = b_1 < b_2 < \dots < b_K < b_{K+1} = u(y),$$

such that every closed interval $[b_k, b_{k+1}]$, $k = 1, \dots, K$ is contained in some element of $\{I_a\}_{a \in A'}$. Now, for every $k = 1, \dots, K$, let I_{a_k} , $a_k \in A'$ be any sub-cover element such that $[b_k, b_{k+1}] \subset I_{a_k}$. Thus there exist $t_k^- < t_k^+$ such that:

$$\varphi_{t_k^-}(z_{a_k}) \sim z_{b_k} \quad \text{and} \quad \varphi_{t_k^+}(z_{a_k}) \sim z_{b_{k+1}},$$

and hence:

$$\varphi_{t_k^+ - t_k^-}(\varphi_{t_k^-}(z_{a_k})) \sim z_{b_{k+1}}.$$

In particular, by Lemma 1, the left-hand term is well-defined. Then, defining:

$$x_k = \varphi_{t_k^-}(z_{a_k}), \quad \text{and} \quad t_k = t_k^+ - t_k^-,$$

we have:

$$\varphi_{t_k}(x_k) = \varphi_{t_k^+ - t_k^-}(\varphi_{t_k^-}(z_{a_k})) \sim z_{b_{k+1}} \sim \varphi_{t_{k+1}^-}(z_{a_{k+1}}) = x_{k+1},$$

as desired. Now, by maximality of $(\mathcal{D}, \varphi)$, if $(t, z) \in \mathcal{D}$, then $(-t, \varphi_t(z)) \in \mathcal{D}$ by [Lemma 1](#). Thus, if $x \succ y$, applying the same construction in the opposite direction, and then reversing the transfers, yields the analogous conclusion. Finally, when $(\mathcal{D}, \varphi)$ is a utility unit for u , the last claim follows from (3), as the sum along the chain telescopes. $\square$

F.2 Proof of [Proposition 2](#)

Proof. For sufficiency, it suffices to note that if $(\mathcal{D}, \varphi)$ is a common unit for $(u^1, \dots, u^S)$, then $(\tilde{\mathcal{D}}, \tilde{\varphi})$, where $\tilde{\varphi}_t = \varphi_{t/\alpha}$, and $\tilde{\mathcal{D}}^{(x)} = \alpha \mathcal{D}^{(x)}$, is a unit for $(\alpha u^1 + \beta^1, \dots, \alpha u^S + \beta^S)$.

For necessity, suppose $S \geq 2$, and that for some tuple of increasing, smooth, and critical-point-free functions $f \equiv (f^1, \dots, f^S)$, whenever a profile $(u^1, \dots, u^S)$ is interpersonally comparable, so is $(f^1 \circ u^1, \dots, f^S \circ u^S)$.

Choose global coordinates on $\mathcal{X}$ such that the projection of $\mathcal{X}$ onto the first factor is non-singleton. Define:

$$u^s(x) = x^1 + c^s$$

for all $s = 1, \dots, S$, where each $c_s \in \mathbb{R}$. Then $(u^1, \dots, u^S)$ are interpersonally comparable, with common utility unit given by $(\mathcal{D}, \varphi)$, where $\varphi_t(x) = x + (t, 0, \dots, 0)$ and $\mathcal{D}$ is the maximal domain associated with this flow on $\mathcal{X}$.⁴⁸

Now, if $(f^1 \circ u^1, \dots, f^S \circ u^S)$ are interpersonally comparable, there exists some $(\tilde{\mathcal{D}}, \tilde{\varphi})$ such that:

$$(f^s \circ u^s)(\tilde{\varphi}_t(x)) = (f^s \circ u^s)(x) + t.$$

Differentiating both sides with respect to t and evaluating at $t = 0$ implies:

$$\dot{\tilde{\varphi}} \cdot \nabla u^s = \frac{1}{(f^s)'(u^s)}$$

for every $s = 1, \dots, S$. In particular, since $\nabla u^s = [1, 0, \dots, 0]^\top$, this implies:

$$\dot{\tilde{\varphi}}^1(x) = \frac{1}{(f^s)'(x^1 + c^s)}.$$

⁴⁸See [Lee \(2012\)](#) Theorem 9.12.

But since f preserves interpersonal comparability for any profile, it does so for the profile obtained by simply varying the c^s . Since any profile obtained in this fashion remains comparable, $(f^1)' = \dots = (f^S)' = \alpha$ for some constant α ; since each f^s is strictly increasing, $\alpha > 0$ and $f^s(t) = \alpha t + \beta_s$. $\square$

Supplemental Appendix

G Quasilinear Representation of Comparable Profiles

Perhaps the most natural example of interpersonally comparable utilities are quasilinear profiles. When $\mathcal{X} = \mathbb{R}_{++}^n$, and every utility is of the form:

$$u^s(x) = x^1 + v^s(x^2, \dots, x^n),$$

it is natural to treat utilities as denominated in a common unit.⁴⁹ Here, since the marginal utilities of the numeraire, x^1 , are all normalized to unity, the ‘utils’ of each u^s may be identified with quantities of numeraire. Since these have exogenous, physical meaning, they can serve as an objective measurement unit into which all utility differences can be converted.⁵⁰

Our next result illustrates that quasilinearity provides a kind of normal-form representation for interpersonally comparable profiles. We show a profile admits a common measurement unit if, and only if, there exists a globally-defined, smooth change of coordinates that simultaneously renders each utility quasilinear.⁵¹ This illustrates that the abstract machinery of virtual commodities may simply be viewed as a coordinate-free generalization of classical commodities, allowing us to speak of numeraire goods even outside the setting of traditional consumer theory.

Before formally stating this result, we introduce a mild, technical regularity condition. Suppose $(\mathcal{D}, \varphi)$ is a virtual commodity, and $x, x' \in \mathcal{X}$. Let $x \sim_\varphi x'$ denote that x and x' are equal, up to some quantity of additional commodity, that is, $x' = \varphi_t(x)$ for some $t \in \mathcal{D}^{(x)}$ or vice versa. We say $(\mathcal{D}, \varphi)$ is a **regular** virtual commodity if the binary relation $\sim_\varphi$ has a closed graph.

⁴⁹In the language of virtual commodities, this corresponds to the virtual numeraire with $\mathcal{D}^{(x)} = (-x^1, \infty)$, for each $x \in \mathcal{X}$, and $\varphi_t(x) = x + (t, 0, \dots, 0)$.

⁵⁰For example, in a quasilinear profile, this amounts to judging that individual s values changing the social allocation from x to x' more than does s' , because their numeraire-denominated willingness-to-pay to induce such a shift exceeds that of s' .

⁵¹By smooth change of coordinates, we mean a smooth bijection with smooth inverse.

Theorem 5. *A regular virtual commodity $(\mathcal{D}, \varphi)$ is a common utility unit for the profile $(u^1, \dots, u^S)$ if, and only if, there exists a smooth manifold $\mathcal{Y}$, open subset $\Omega \subseteq \mathbb{R} \times \mathcal{Y}$, and smooth change-of-coordinates $h : \mathcal{X} \rightarrow \Omega$, such that:*

(i) *For all $(x, t) \in \mathcal{D}$,*

$$h(\varphi_t(x)) = h(x) + (t, 0, \dots, 0).$$

(ii) *For all $s \in \mathcal{S}$ and $(t, y) \in \Omega$,*

$$u^s(h^{-1}(t, y)) = t + v^s(y).$$

[Theorem 5](#) states that whenever $(u^1, \dots, u^S)$ are interpersonally comparable, they are jointly quasilinear under some change of coordinates. The first claim states that this change of variables transforms the virtual numeraire into the new, horizontal-axis commodity. This allows us to interpret the first component of vectors in Ω as reflecting the level of virtual commodity contained in an allocation. The second claim states that under this change of variables, each utility u^s is quasilinear with respect to this commodity.

Example 6. Suppose $\mathcal{X} = \mathbb{R}_{++}^2$, and suppose $(u^1, \dots, u^S)$ are all of the form:

$$u^s(x) = \ln \tilde{u}^s(x),$$

where each $\tilde{u}^s$ is strictly increasing, and homogeneous of degree one.^{[52](#)} Define the virtual commodity $\mathcal{D} = \mathcal{X} \times \mathbb{R}$ and $\varphi_t(x) = e^t x$, i.e. where φ_t scales $x \in \mathcal{X}$ proportionally, by the scalar e^t . It follows immediately that:

$$u^s(\varphi_t(x)) = \ln \tilde{u}^s(e^t x) = \ln(e^t \tilde{u}^s(x)) = u^s(x) + t,$$

and hence $(u^1, \dots, u^S)$ is interpersonally comparable. Consider now the change of coordinates h mapping (x^1, x^2) to the modified polar coordinates $(\ln r, \theta)$. Here, $\Omega = \mathbb{R} \times (0, \frac{\pi}{2})$ and, in these coordinates:

$$u^s(h^{-1}(\ln r, \theta)) = \ln \tilde{u}^s(r \cos \theta, r \sin \theta) = \ln r + u^s(\cos \theta, \sin \theta)$$

is quasilinear in first coordinate, $\ln r$, as desired. ■

⁵²Any such function $\tilde{u}^s$ is necessarily positive-valued, and hence u^s is well-defined.

G.1 Proof of Theorem 5

Proof. ($\implies$): Suppose that $(\mathcal{D}, \varphi)$ is a virtual commodity which is a common utility unit for a profile $(u^1, \dots, u^S)$, and that $\sim_\varphi$ is a closed binary relation. First, we note $\sim_\varphi$ is an equivalence relation. It is defined symmetrically, and since for any $x \in \mathcal{X}$, $0 \in \mathcal{D}^{(x)}$ and $\varphi_0(x) = x$, it is reflexive. Suppose then that $x \sim_\varphi y$ and $y \sim_\varphi z$. If $\varphi_t(x) = y$ and $\varphi_{t'}(y) = z$, then $\varphi_{t'+t}(x)$ is well-defined and equals z . Thus suppose $\varphi_t(x) = y$ and $\varphi_{t'}(z) = y$. Then $\varphi_{-t}(y) = x$ and hence $\varphi_{t'-t}(z)$ is well-defined and equals x . Thus $\sim_\varphi$ is transitive and hence an equivalence relation.

We now show the quotient map $\pi : \mathcal{X} \rightarrow \mathcal{X}/\sim_\varphi$ is open. First, note that for any $t \in \mathbb{R}$, let $\mathcal{D}_t = \{x \in \mathcal{X} : t \in \mathcal{D}^{(x)}\}$. As $\mathcal{D}$ is open, so too is each $\mathcal{D}_t$. Moreover, $\varphi_t : \mathcal{D}_t \rightarrow \mathcal{D}_{-t}$ is a diffeomorphism with inverse φ_{-t} . Now, let $U \subseteq \mathcal{X}$ be open. Then:

$$\begin{aligned} \pi^{-1}(\pi(U)) &= \{x \in \mathcal{X} : \varphi_t(x) \in U \text{ for some } t \in \mathbb{R}\} \\ &= \bigcup_{t \in \mathbb{R}} \varphi_t(U \cap \mathcal{D}_t), \end{aligned}$$

which is open. Thus, in the quotient topology on $\mathcal{X}/\sim_\varphi$, $\pi(U)$ is open for any open $U \subseteq \mathcal{X}$, and hence π is an open map.

We now show that $\mathcal{X}/\sim_\varphi$ is a smooth manifold. First, we verify the quotient is Hausdorff. Let $\pi(x) \neq \pi(x')$. Then $(x, x') \notin \sim_\varphi$ and hence there exist open neighborhoods U of x and V of x' in $\mathcal{X}$ such that $(U \times V) \cap \sim_\varphi = \emptyset$. As both U and V are open, so too are $\pi(U)$ and $\pi(V)$, as π is an open map. Then $\pi(U) \cap \pi(V) = \emptyset$, as their product is disjoint from the graph of $\sim_\varphi$, and thus $\mathcal{X}/\sim_\varphi$ is Hausdorff. Moreover, since $\mathcal{X}$ is second-countable, so too is $\mathcal{X}/\sim_\varphi$, as by openness of π , the image of any countable basis for $\mathcal{X}$ under π forms a countable basis for $\mathcal{X}/\sim_\varphi$.

We now construct a smooth atlas for $\mathcal{X}/\sim_\varphi$. Fix any $s \in \mathcal{S}$, and let $x \in \mathcal{X}$. Let $\mathcal{I}$ denote the u^s -indifference surface through x . Because u^s is critical-point-free, $\mathcal{I}$ is an embedded smooth submanifold of $\mathcal{X}$, e.g. Lee (2012) Corollary 5.14. Now, let $U \subset \mathcal{I}$ be a coordinate neighborhood of x on this submanifold. Define:

$$\mathcal{D}_U = \{(x', t') \in U \times \mathbb{R} : (x', t') \in \mathcal{D}\},$$

and let $\Psi : \mathcal{D}_U \rightarrow \mathcal{X}$ via $\Psi(x', t') = \varphi_{t'}(x')$. Suppose $\Psi(x', t') = \Psi(x'', t'')$, i.e. $\varphi_{t'}(x') = \varphi_{t''}(x'')$. Then $u^s(x') = u^s(x'')$, hence by (3), $u(\varphi_{t'}(x')) - u^s(\varphi_{t''}(x'')) =$

$t' - t''$ and thus $t' = t''$. But as $\varphi_{t'} : \mathcal{D}_{t'} \rightarrow \mathcal{D}_{-t'}$ is injective, we obtain $x' = x''$. Thus Ψ is itself injective.

Define $X(x) = \frac{\partial}{\partial t} \varphi_t(x) \big|_{t=0}$. Fix some $(x', t') \in \mathcal{D}_U$, and let $x'' = \varphi_{t'}(x')$. Since U is open in $\mathcal{I}$, a vector ξ is tangent to $\mathcal{U}$ at x' if and only if $\xi \in \ker du_{x'}$. Then for any $\xi \in T_{x'}U$ and $\lambda \in \mathbb{R}$,

$$d\Psi_{(x', t')}(\xi, \lambda) = d(\varphi_{t'})_{x'}(\xi) + \lambda X_{x''}. \quad (13)$$

These two terms belong to complementary subspaces, as $\langle X, \nabla u \rangle = 1$ implies $X_{x''} \notin \ker du_{x''}$ and hence is not tangent to U . Now, differentiating both sides of (3) with respect to x and evaluating at (x', t') yields:

$$du_{x''} \circ d(\varphi_{t'})_{x'} = du_{x'},$$

and hence $d(\varphi_{t'})_{x'}(T_{x'}U) \subseteq \ker du_{x''}$; as $d(\varphi_{t'})_{x'}$ is an isomorphism, these two subspaces have equal dimension, and hence coincide. Now, suppose:

$$d\Psi_{(x', t')}(\xi, \lambda) = 0.$$

Applying $du_{x''}$ to both sides yields:

$$0 = du_{x''} \left(d(\varphi_{t'})_{x'}(\xi) + \lambda X_{x''} \right) = \lambda$$

and hence $\lambda = 0$. Plugging this into (13), this implies $0 = d(\varphi_{t'})_{x'}(\xi)$. But as noted above, $d(\varphi_{t'})_{x'}$ is an isomorphism hence $\xi = 0$. Thus $d\Psi_{(x', t')}$ is an isomorphism and therefore Ψ is a local diffeomorphism.

Now, by definition, the range $\Psi(\mathcal{D}_U) = \pi^{-1}(\pi(U))$, hence $\pi^{-1}(\pi(U))$ is open, and therefore Ψ is a diffeomorphism. But this implies that $\pi(U)$ is an open subset of $X/\sim_\varphi$, and that $\pi|_U \rightarrow \pi(U)$ is a homeomorphism onto an open subset. As U is a coordinate neighborhood, this yields charts covering $X/\sim_\varphi$.

Suppose $U, V \subset X/\sim_\varphi$ are two coordinate neighborhoods such that $U \cap V \neq \emptyset$. Then for some open subset $U' \subseteq \{x \in \mathcal{X} : u^s(x) = c\}$ and $V' \subseteq \{x \in \mathcal{X} : u^s(x) = d\}$, the transition map is simply the restriction of φ_{d-c} , which is smooth. Hence these charts define a smooth $(d-1)$ -manifold structure on $X/\sim_\varphi$.

We now are in position to define h , $\mathcal{Y}$, and Ω . Let $\mathcal{Y} = X/\sim_\varphi$, and define $h : \mathcal{X} \rightarrow \mathbb{R} \times \mathcal{Y}$ via:

$$h(x) = (u^s(x), \pi(x)).$$

The map h is injective, as $h(x) = h(x')$ implies $\pi(x) = \pi(x')$ and hence $\varphi_t(x) = x'$. But since $u^s(x) = u^s(x')$, by (3), we must have $x = x'$. Suppose now $dh_x(\zeta) = 0$ for some tangent vector $\zeta \in T_x\mathcal{X}$. Since $dh_x = [du_x^s \ d\pi_x]^\top$, this implies $d\pi_x(v) = 0$ hence v is proportional to X_x . But since $du_x^s(\lambda X_x) = \lambda$, it follows $\lambda = 0$, and hence $v = 0$. Thus dh_x is injective and since both the domain and codomain are d -dimensional, dh_x is an isomorphism, making h a local diffeomorphism. Because we have already shown h to be injective, and both $\mathcal{X}$ and $\mathbb{R} \times X / \sim_\varphi$ are d -dimensional manifolds, $\mathcal{X}$ without boundary, we obtain that h is a diffeomorphism from $\mathcal{X}$ onto the open set $h(\mathcal{X}) \subseteq \mathbb{R} \times X / \sim_\varphi$ (e.g. Prop. 4.22.(d) in Lee 2012). We then define $\Omega = h(\mathcal{X}) \subseteq \mathbb{R} \times X / \sim_\varphi$.

Finally, we now verify (i) and (ii). For (i), note that for all $(x, t) \in \mathcal{D}$,

$$\begin{aligned} h(\varphi_t(x)) &= (u^s(\varphi_t(x)), \pi(\varphi_t(x))) \\ &= (u^s(x) + t, \pi(x)) \\ &= h(x) + (t, 0, \dots, 0), \end{aligned}$$

where the second equality follows from the fact u^s satisfies (3). For (ii), note that from the above, $(h_*X)_{(t,y)} = dh_{h^{-1}(t,y)}(X_{h^{-1}(t,y)}) = \partial_t$. Thus if $h(x) = (t, y)$, then for any $s' \in \mathcal{S}$:

$$\frac{\partial}{\partial t} u^{s'}(h^{-1}(t, y)) = \langle X, \nabla u \rangle|_x = 1.$$

Writing $\Omega = h(\mathcal{X})$, let $\Omega^{(y)} = \{t : (t, y) \in \Omega\}$. For any y , this is a connected interval, as if $h(x) = (t, y)$, this just equals $u^s(\pi^{-1}(\pi(x))) = u^s(\varphi_{\mathcal{D}^{(x)}}(x))$, which is connected as $\mathcal{D}^{(x)}$ is. As each $\Omega^{(y)}$ is connected, $u^{s'}(h^{-1}(t, y)) - t$ is then constant on $\Omega^{(y)}$ and hence depends only on y as a function defined on all Ω . Define this function to be $v^{s'}(y) = u^{s'}(h^{-1}(t, y)) - t$. Then by identity, $u^{s'}(h^{-1}(t, y)) = t + v^{s'}(y)$, as desired.

($\Leftarrow$): Suppose now (i) and (ii) hold. Suppose $h(x) = (t, y)$ and let $t' \in \mathcal{D}^{(x)}$. Then:

$$\begin{aligned} u^s(x) &= u^s(h^{-1}(t, y)) \\ &= u^s(h^{-1}(t + t', y)) - t' \\ &= u^s(\varphi_{t'}(h^{-1}(t, y))) - t' \\ &= u^s(\varphi_{t'}(x)) - t', \end{aligned}$$

where the first equality is an identity, the second follows from (ii), the third from (i), and the fourth again is an identity. Hence u^s satisfies (3) for each $s \in \mathcal{S}$. $\square$

H Locally Comparable Representations

For ease of exposition, we will illustrate our results in the case of full-rank profiles, containing $S = d + 1$ individuals.⁵³ Consider, then, a fixed status quo allocation x_0 , and suppose the planner observes a profile $(\succsim^1, \dots, \succsim^{d+1})$. Let $(u^1, \dots, u^{d+1})$ denote an arbitrary profile of representations. We will make use of the following regularity hypothesis.

(CR.1) **Full Profile Rank:** The subprofile $(u^1, \dots, u^d)$ has constant and full rank about x_0 , i.e. the matrix:

$$\mathcal{J}(x) = \begin{bmatrix} \vdots & & \vdots \\ \nabla u^1 & \dots & \nabla u^d \\ \vdots & & \vdots \end{bmatrix}^\top \text{ is invertible} \quad (14)$$

in a neighborhood of x_0 . In addition, we will suppose:

$$\nabla u^{d+1}(x_0) = \sum_{i=1}^d c_i \nabla u^i(x_0),$$

for positive scalars $c_1, \dots, c_d > 0$.

Assumption (CR.1) requires the gradients of some d -individual subprofile not all lie on some lower dimensional surface. We also require that the final gradient does not belong to the span of any proper sub-collection of these. However, the assumption that it is specifically a *positive* linear combination is purely for notational convenience. Finally, to avoid a proliferation of indices, we will simply write $\bar{u}$ for u^{d+1} .

Suppose now (CR.1) holds, and let $\mathcal{N}_0$ denote any neighborhood of x_0 on which $\mathcal{J}$ is invertible, and the gradient condition on $\bar{u}$ remains valid. Given any smooth function $f : \mathcal{N}_0 \rightarrow \mathbb{R}$, we define its **utility derivatives**, for any $i = 1, \dots, d$, via:

$$f_i \equiv \frac{\partial f}{\partial u^i} = \sum_{j=1}^d \frac{\partial f}{\partial x^j} (\mathcal{J}^{-1})_{ji},$$

These quantities are simply the derivatives of f , instead computed with respect to the *utility* coordinates $u \equiv (u^1, \dots, u^d)$.

⁵³All our results extend straightforwardly, albeit at the cost of more cumbersome notation, to profiles of arbitrary, finite size, as well as to profiles of lower rank; see [Section H.3](#).

This perspective allows us to consider the final utility $\bar{u}$ as a function $\bar{u}(u)$ of the utilities of the remaining members of society. Doing so, we define the **separability indices** of $\bar{u}$ via:

$$\underbrace{\frac{\bar{u}_{ij}}{\bar{u}_i \bar{u}_j} - \frac{\bar{u}_{ik}}{\bar{u}_i \bar{u}_k}, \quad i, j, k \text{ distinct}}_{\text{If } d \geq 3} \quad \text{or} \quad \underbrace{\sqrt{\left| \frac{1}{\bar{u}_1 \bar{u}_2} \partial_{12} \ln \left(\frac{\bar{u}_1}{\bar{u}_2} \right) \right|}}_{\text{If } d = 2},$$

where all subscripts refer to utility derivatives. Let $\mathbb{H}$ denote the set of such indices.⁵⁴ It may be verified that, given $\bar{u}$, each $h \in \mathbb{H}$ is independent of the choices of representation $u^1, \dots, u^d$. Moreover, while in general the value of a given $h \in \mathbb{H}$ depends on the particular choice of representation $\bar{u}$, the property that any given $h = 0$ is purely ordinal.

As we will shortly show, the indices in $\mathbb{H}$ may be regarded as quantifying how close $\bar{u}$ is, up to a monotone transform, to being an additively separable function of the utilities $u^1, \dots, u^d$. Equivalently, they reflect how close the utility-derivative marginal rates of substitution $\bar{u}_i/\bar{u}_j$ are to being independent of the utility levels of the rest of society $k \neq i, j$.

Before providing further interpretation, we first introduce our next regularity condition, which rules out singularities in the system $\mathbb{H}$ about $u_0 = u(x_0)$.

(CR.2) **Constant Index Rank:** For all $h \in \mathbb{H}$, $\text{sign}(h(u))$ is constant in a neighborhood of u_0 .⁵⁵

Under (CR.2), there are two possibilities. Either every function in $\mathbb{H}$ is uniformly zero on some neighborhood of x_0 , or there exists some $h \in \mathbb{H}$ such that $h(u_0)$ is locally strictly positive.⁵⁶ These two cases lead to qualitatively different behavior, hence we treat them separately.

⁵⁴That is, when $d \geq 3$, we let $\mathbb{H}$ denote the set of all functions $\frac{\bar{u}_{ij}}{\bar{u}_i \bar{u}_j} - \frac{\bar{u}_{ik}}{\bar{u}_i \bar{u}_k}$ for distinct i, j , and k . If instead $d = 2$, $\mathbb{H}$ denotes the singleton family, containing only $\sqrt{\left| \frac{1}{\bar{u}_1 \bar{u}_2} \partial_{12} \ln \left(\frac{\bar{u}_1}{\bar{u}_2} \right) \right|}$.

⁵⁵Recall $\text{sign}(t) = 1$ if $t > 0$, -1 if $t < 0$, and 0 if $t = 0$.

⁵⁶When $d \geq 3$, the functions corresponding to the tuple (i, j, k) and (i, k, j) have opposite sign. As a consequence if any $h \in \mathbb{H}$ is non-zero at u_0 , at least one will be strictly positive.

H.1 The Separable Case

Suppose first every $h \in \mathbb{H}$ vanishes on some neighborhood of u_0 . By possibly restricting further, we may suppose this is true on some $\mathcal{N} \subseteq u(\mathcal{N}_0)$. In this case, there exist utility representations $f^1 \circ u^1, \dots, f^d \circ u^d$ for $\succsim^1|_{u^{-1}(\mathcal{N})}, \dots, \succsim^d|_{u^{-1}(\mathcal{N})}$, and a smooth, strictly increasing, and critical-point-free $\bar{f} : \bar{u}(\mathcal{N}) \rightarrow \mathbb{R}$, such that:

$$(\bar{f} \circ \bar{u})(u) = \sum_{i=1}^d f^i(u^i). \quad (15)$$

Let $v^i = f^i \circ u^i$, for each $i = 1, \dots, d$. We term the tuple $v \equiv (v^1, \dots, v^d)$ a **locally separable representation** for the profile $(\succsim^1, \dots, \succsim^{d+1})$.⁵⁷ Thus this case corresponds precisely to the locally separable setting treated in the main text.

H.2 The Non-separable Case

Suppose now that $\mathbb{H}$ contains some function $\bar{h}$ which is strictly positive at x_0 . In this case, nearby $u_0 = u(x_0)$, we can define the functions:

$$K^i(u) = -\ln(\bar{h}(u) \bar{u}_i(u)),$$

for $i = 1, \dots, d$. In light of (CR.1) and (CR.2), these are well-defined. As the following lemma shows, the functions $K^1, \dots, K^d$ generate an overdetermined system of partial differential equations, which characterize when a local virtual commodity is numeraire, for every preference in $(\succsim^1, \dots, \succsim^{d+1})$.

Proposition 3. *Assume (CR.1) and (CR.2), and let $\bar{h} \in \mathbb{H}$ be positive at u_0 . Suppose $(\mathcal{D}, \varphi)$ is a local virtual commodity defined about u_0 , and in u -coordinates, let.*⁵⁸

$$\dot{\varphi}(u) = F(u) = \begin{bmatrix} F^1(u) \\ \vdots \\ F^d(u) \end{bmatrix}.$$

Then $(\mathcal{D}, \varphi)$ satisfies (N.1) and (N.2) for every preference in $(\succsim^1, \dots, \succsim^{d+1})$, if, and

⁵⁷The tuple of representations $v = (v^1, \dots, v^d)$ is unique up to a component-wise transform of the form $v^i \mapsto \alpha v^i + \beta^i$, where the common $\alpha > 0$, and $\beta^i \in \mathbb{R}$.

⁵⁸In particular, F^i is the component of F pointing in direction ∇u^i .

only if, each $F^i > 0$, and F satisfies:

$$\begin{bmatrix} F_1^1 & \cdots & F_d^1 \\ \vdots & \ddots & \vdots \\ F_1^d & \cdots & F_d^d \end{bmatrix} = \begin{bmatrix} \langle F, \nabla K^1 \rangle & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & \langle F, \nabla K^d \rangle \end{bmatrix}. \quad (16)$$

[Proposition 3](#) shows that the d components of the generator of a virtual numeraire must satisfy an overdetermined system of d^2 partial differential equations. Informally, the equations (16) state each transformation φ_t must preserve all local, geometric information about every indifference curve, such as the utility-derivative marginal rates of substitution of $\bar{u}$, and any structural relations between them. This may be viewed as a generalization of the classical fact that quasilinear preferences are characterized by the independence of their marginal rates of substitution on the level of numeraire.⁵⁹

In general, no non-zero solution to (16) will exist, unless the system satisfies strong compatibility, or ‘integrability’ conditions. For any distinct $i, j_0 = 1, \dots, d$, let $Q_{j_0}^i = K_{j_0}^i$, and for any tuple of indices $(i, j_0, \dots, j_L)$ with $i \neq j_0$ and $L \geq 0$, recursively define:

$$Q_{j_0 \dots j_{L+1}}^i = \frac{\partial Q_{j_0 \dots j_L}^i}{\partial u^{j_{L+1}}} + Q_{j_0 \dots j_L}^i \sum_{l=0}^L K_{j_{L+1}}^{j_l}.$$

Let $\mathcal{M}_L$ denote the matrix of functions whose rows are given by the vectors:

$$\left[\nabla Q_{j_0 \dots j_{L'}}^i + Q_{j_0 \dots j_{L'}}^i \sum_{l=0}^{L'} \nabla K^{j_l} \right]^\top, \quad (17)$$

for each tuple $(i, j_0, \dots, j_{L'})$ with $i \neq j_0$ and $L' \leq L$. Since the rank of any $\mathcal{M}_L$ cannot exceed d , there is necessarily some unique, smallest L^* such that, evaluated at x_0 , the ranks of $\mathcal{M}_{L^*}$ and $\mathcal{M}_{L^*+1}$ coincide. Our final regularity condition requires the ranks of these matrices to be locally constant.

(CR.3) **Constant System Rank:** For each $0 \leq L \leq L^* + 1$, the ranks of the matrices $\mathcal{M}_L$ are locally constant about x_0 .

⁵⁹See [Blackorby et al. \(1978\)](#) and [Sono \(1961\)](#). See also [Theorem 5](#).

We say that the system (16) satisfies the **compatibility conditions** if $\ker(\mathcal{M}_{L^*})$, evaluated at x_0 , contains a strictly positive vector. Our next theorem not only characterizes the existence of common, local virtual numeraire, but also shows that the set of such commodities has a well-defined dimension, and is smoothly parameterized by at most $d = \dim(\mathcal{X})$ parameters.

Theorem 6. *Assume (CR.1) - (CR.3), and suppose $\mathbb{H}$ contains some locally non-zero function. Then the set of local virtual numeraire for $(\succsim^1, \dots, \succsim^{d+1})$ is non-empty if, and only if the compatibility conditions:*

$$\ker(\mathcal{M}_{L^*}) \cap \mathbb{R}_{++}^d \neq \emptyset$$

hold at x_0 . When this is true, the set of local virtual numeraire is non-empty, and depends smoothly on $d - \text{rank}(\mathcal{M}_{L^})$ parameters.*

Theorem 6 characterizes when a common local virtual numeraire exists for a preference profile. This, in turn, completely describes the set of profiles of comparable representations for these preferences.

This result straightforwardly extends to profiles where $S \neq d + 1$, as well as to various cases where (CR.1) does not hold, such as when a profile's rank is less than d , or ∇u^{d+1} is not a positive linear combination of other gradients.⁶⁰

Together Theorem 4 and Theorem 6 show that, when non-empty, the set of common, local virtual numeraire is always finite dimensional, and of dimension at most equal to $d + 1$. Thus, interpersonal comparability presents an exacting requirement. For any sufficiently rich profile, of the infinite-dimensional family of possible local utility representations consistent with the given preferences, only a finite-dimensional subset will be interpersonally comparable.

Theorem 6 not only allows us to describe the set of common, virtual numeraire for a given profile, but also to obtain necessary and sufficient conditions on *families* of profiles for the existence of comparable representations. We illustrate this below.

Example 7. Suppose that $\mathcal{X} = \mathbb{R}_{++}^3$, and consider the preference profile represented by the utilities $u^i(x) = x^i$, for $i = 1, 2, 3$, and $\bar{u}(x) = g(x^1 + x^2) + x^3$ for some

⁶⁰See Section H.3.

increasing, strictly concave g .⁶¹ Note that varying g generally does not correspond to applying any monotone transform to $\bar{u}$, hence we may regard g as indexing a family of preference profiles.

We claim any tuple of preferences of this form admits a common, local virtual numeraire about an allocation x_0 if, and only if, restricted to some neighborhood of x_0 , g is the Bernoulli utility of a hyperbolic expected utility preference (Merton 1971), i.e. it satisfies:

$$-\frac{g'(s)}{g''(s)} = a + bs, \quad (18)$$

for constants $a, b \in \mathbb{R}$.⁶² For any such g , we will show the set of common, local numeraires depends smoothly on three parameters.⁶³

To prove this, we will make use of Proposition 3 and Theorem 6. Firstly, we seek a positive function in $\mathbb{H}$. Letting $s = x^1 + x^2$, since $\bar{u}_1 = \bar{u}_2 = g'(s)$, and $\bar{u}_3 = 1$, the only functions in $\mathbb{H}$ that are positive at any given point are $\bar{h} = -g''(s)/g'(s)^2$. Thus we obtain:

$$K^1 = K^2 = -\ln \frac{-g''(s)}{g'(s)}, \quad \text{and} \quad K^3 = -\ln \frac{-g''(s)}{g'(s)^2}.$$

By Proposition 3, it suffices to instead determine whether any vector field F , defined in a neighborhood of x_0 , satisfies (16) for these K .

Since $K^1 = K^2$, by (16) it follows any F must satisfy $F_1^1 = F_2^2$. However, since (16) also implies each F_i^i must be a function solely of x^i , it follows that $F_1^1 = F_2^2 = \alpha$, for some constant α . Integrating, we obtain $F^i = \alpha x^i + \beta^i$, for $i = 1, 2$.

Plugging these functional forms for F^1 and F^2 back into (16) yields an ordinary differential equation for g that must be satisfied for any such X to exist:

$$\frac{T'(s)}{T(s)} = \frac{\alpha}{\alpha s + \beta},$$

⁶¹Given our choices of u^1, u^2 , and u^3 , we may think of $\mathcal{X}$ as already being in utility coordinates. Thus, in particular, the utility derivatives of $\bar{u}$ coincide with its partial derivatives with respect to each x^i .

⁶²Examples include both log or power functions (the CRRA family) and exponential (CARA) utilities. Since g is assumed to be increasing and strictly concave, $a + bs$ is strictly positive on the range of values taken by $x^1 + x^2$ about x_0 .

⁶³If, instead, we required g to be only weakly concave, on regions where g is linear, Theorem 4 would apply and instead yield four-parameter families of local solutions.

where $T(s) = -g'(s)/g''(s)$. In the context of risk, this quantity would correspond to the Arrow-Pratt measure of risk tolerance for the expected utility preference with Bernoulli utility g . Integrating both sides yields:

$$-\frac{g'(s)}{g''(s)} = as + b,$$

as desired. Thus for any common virtual numeraire to exist, g must necessarily obey (18).

Conversely, to show this is not only necessary but also sufficient for the existence of some common, local numeraire, we apply Theorem 6. By direct calculation, satisfaction of (18) implies each vector of the form (17) is uniformly zero. Thus $L^* = 0$, and $\mathcal{M}_{L^*}$ is simply the zero matrix, and hence satisfies the integrability conditions. As a consequence, Theorem 6 guarantees the set of common, local virtual numeraires is non-empty, and depends smoothly on 3 parameters, as desired. ■

H.3 Theorem 6 for General Profiles

H.3.1 Underdetermined Profiles: Rank = S

Consider a preference profile $(\succsim^1, \dots, \succsim^S)$ whose rank (i.e. the rank of the $d \times S$ matrix whose columns are the gradients ∇u^s of arbitrary choices of representations) is equal to S . In this case, given any choice of representations $u^1, \dots, u^S$, the gradients of these utilities are linearly independent at x_0 ; we will assume that this rank condition holds locally in some neighborhood of x_0 .

In this case, the profile admits an infinite-dimensional family of virtual numeraires. To see this, given any choice of profile of representations $(u^1, \dots, u^S)$, by linear independence of $\nabla u^1, \dots, \nabla u^S$, their affine hull cannot contain the origin, thus by Theorem 3 there exists a common utility unit for $(u^1, \dots, u^S)$. Since the independence of gradients at x_0 is preserved under monotone transformation, this will be true of any representing profile, yielding, by Theorem 1, a generally different unit for each other choice of utilities.

H.3.2 Overdetermined Profiles: $S > \text{Rank} + 1$

We first note that in such cases, it is without loss of generality to assume the profile rank r is equal to d . If not, by the Rank Theorem (Lee 2012 Theorem 4.12), there

exist local coordinates about x_0 of the form $(u^1, \dots, u^r, z^{r+1}, \dots, z^d)$. Thus any local numeraire defined on the r -dimensional slice, spanned by the utility coordinates, through x_0 , can be turned into a full local solution simply by defining it to be locally constant in the z variables.

Thus, without loss of generality, we suppose the profile rank equals d . While both [Proposition 3](#) and the [Veblen and Thomas \(1926\)](#) integrability theorem can be extended to the case of arbitrarily more than $d + 1$ preferences (see discussion in [Eisenhart 1927](#)), doing so directly yields additional local rank conditions which must be satisfied, and additional, increasingly complex integrability conditions. Instead, a far simpler practical approach is to simply apply [Theorem 4](#) or [Theorem 6](#) to any subprofile of $d + 1$ preferences, hence obtaining a complete description of the subprofiles common virtual numeraires, depending on at most $d + 1$ free parameters. For each subsequent preference $d + 2$ onward, [Lemma 5](#) gives a simple necessary and sufficient condition to check whether a given φ indeed is a virtual numeraire.

Concretely, [Lemma 5](#) states that if F is a solution to [\(16\)](#), then, for any choice of utility $\tilde{u}$ for any additional preference, that F is the generator of a local virtual numeraire for the underlying preferences if and only if (i) $\langle F, \nabla \tilde{u} \rangle$ can be written as $h \circ \tilde{u}$, and (ii) $\langle F, \nabla \tilde{u} \rangle > 0$. Condition (i) may be seen as a generalization of Euler's homogeneous function theorem.⁶⁴ Thus, in practice, by simply computing the directional derivative $\langle F, \nabla \tilde{u} \rangle$ as a function of the parameters given by [Theorem 6](#) or [Theorem 4](#) and checking conditions (i) and (ii), one obtains a simple method of extending these results to larger profiles.

H.4 Proofs

H.4.1 Preliminary Lemmas

In all that follows, $f^*\omega$ denotes the pullback of the 1-form ω under the smooth function f . We say two 1-forms are proportional if they differ by a positive, smooth, multiple.

Lemma 3. *Let $\succsim$ be a preference on $\mathcal{X}$ with utility u , and $U, V \subseteq \mathcal{X}$ connected open sets, where the restriction of every indifference surface of $\succsim$ to U is connected.*

⁶⁴Euler's homogeneous function theorem (e.g. [Mas-Colell et al. 1995](#)) corresponds to the special case where F is the vector field $[x^1, \dots, x^d]^\top$, which is the generator of the virtual commodity $\varphi_t(x) = e^t x$, which is itself a numeraire for any monotone, homothetic preference.

Then for any diffeomorphism $\phi : U \rightarrow V$, and any 1-form ω proportional to du , the following are equivalent:

- (i) For all $x, x' \in U$, $x \succsim x'$ if and only if $\phi(x) \succsim \phi(x')$; and
- (ii) For some smooth $g : U \rightarrow \mathbb{R}_{++}$, we have $\phi^*\omega = g\omega$.

Proof. Since $\phi : U \rightarrow V$ is a diffeomorphism, (i) is equivalent to u and $u \circ \phi$ representing the same preference on U . Thus (i) is equivalent to the existence of some strictly increasing map $f : u(U) \rightarrow \mathbb{R}$ such that $u \circ \phi = f \circ u$ on U .

We now claim f is smooth. Pick any point $x \in U$. As u is critical-point-free, by the Rank Theorem (Lee 2012 Theorem 4.12), there exists some neighborhood $x \in W \subseteq U$, open interval $I \subseteq \mathbb{R}$, open ball $B \subseteq \mathbb{R}^{d-1}$, and diffeomorphism $h : I \times B \rightarrow W$ such that $u(h(t, b)) = t$. Thus, $(u \circ \phi \circ h)(t, b)$ is smooth, and since u is constant in b , and $u \circ \phi$ represents the same preference, $(u \circ \phi \circ h)(t, b)$ is constant in b . Thus f is simply the restriction of $u \circ \phi \circ h$ to any slice of the form $I \times \{b\}$, and hence smooth. Since smoothness is local, this implies f is smooth on $u(U)$.

Now, differentiating $u \circ \phi = f \circ u$ at any $x \in W$, we obtain $du_x d\phi_x = f'(u(x)) du_x$. Since ϕ is a diffeomorphism, $d\phi$ is invertible; since u is critical-point-free, $du \neq 0$, hence $f'(u(x)) \neq 0$; since f is increasing, $f'(u(x)) > 0$. Thus (i) is equivalent to $u \circ \phi = f \circ u$ for some strictly increasing, critical-point-free smooth map f .

Now, suppose $u \circ \phi = f \circ u$ for such an f , and let $\omega|_U = a du$ for some smooth $a : U \rightarrow \mathbb{R}_{++}$, and likewise $\omega|_V = b du$ for smooth $b : V \rightarrow \mathbb{R}_{++}$. Then by direct calculation:

$$\phi^*\omega = \phi^*(b du) = (b \circ \phi) d(u \circ \phi) = (b \circ \phi) d(f \circ u) = \frac{(b \circ \phi) f'(u)}{a} \omega,$$

and hence (i) implies (ii). Conversely, suppose $\phi^*\omega = (b \circ \phi) d(u \circ \phi) = ga du$, where everywhere $g > 0$. Let $x \sim y$. Since the indifference sets of $\succsim$ are connected, there exists some smooth path $\gamma : [0, 1] \rightarrow U$ such that $\gamma(0) = x$, $\gamma(1) = y$, and γ is contained in the indifference surface through x and y . Since for all t , $\gamma'(t) \in \ker du = \ker d(u \circ \phi)$, by the Gradient Theorem, $u \circ \phi$ is constant on the level sets of u , thus globally we obtain some function $f : u(U) \rightarrow \mathbb{R}$ such that $f \circ u = u \circ \phi$. By an identical argument to before, f is smooth; since $u \circ \phi = f \circ u$, $f'(u) du = d(f \circ u) = d(u \circ \phi) = \frac{ga}{b \circ \phi} du$, and hence $a, b, g > 0$ implies $f'(u) > 0$. Therefore f is strictly increasing and critical-point-free and thus (ii) implies (i). $\square$

Now, suppose that (CR.1) holds for the profile $(\succsim^1, \dots, \succsim^{d+1})$. Let $u^1, \dots, u^d$, and $\bar{u}$ denote arbitrary choices of representation for each preference. Then the 1-form:

$$d\bar{u} = \sum_{i=1}^d \bar{u}_i du^i,$$

for positive-valued utility derivatives $\bar{u}_1, \dots, \bar{u}_d$. (Throughout, we do not employ Einstein summation convention.) For each $i = 1, \dots, d$, define:

$$\omega^i = \bar{u}_i du^i,$$

$\bar{u}_i \equiv \partial/\partial u^i \bar{u}$. By (CR.1) these one-forms define a local coframe near x_0 .

Lemma 4. *Let $U, V \subseteq \mathcal{X}$ be connected open sets such that the restriction of each indifference surface of $\succsim^1, \dots, \succsim^{d+1}$ to U is connected, and (CR.1) holds on U and V . Then for any diffeomorphism $\phi : U \rightarrow V$, the following are equivalent:*

- (i) *For all $x, x' \in U$, we have $x \succsim^i x'$ if and only if $\phi(x) \succsim^i \phi(x')$, for all $i = 1, \dots, d+1$; and*
- (ii) *For some common, smooth $g : U \rightarrow \mathbb{R}_{++}$, we have $\phi^* \omega^i = g \omega^i$ for all $i = 1, \dots, d$.*

Proof. By Lemma 3, there exist smooth functions $g^i : U \rightarrow \mathbb{R}_{++}$ such that, on U , $\phi^* \omega^i = g^i \omega^i$ and (N.1) for $\succsim^i$ coincide. Similarly, there exists $\bar{g} : U \rightarrow \mathbb{R}_{++}$ such that $\phi^* d\bar{u} = \bar{g} d\bar{u}$ is equivalent to (N.1) for the restriction of $\succsim^{d+1}$ to U . Thus, on U , (i) implies:

$$\begin{aligned} \phi^*(d\bar{u}) &= \phi^* \left[\sum_{i=1}^d \bar{u}_i du^i \right] \\ &= \sum_{i=1}^d g^i \omega^i, \end{aligned}$$

where, by definition, $d\bar{u} = \sum_i \omega^i$. But since we also have $\phi^* d\bar{u} = \bar{g} d\bar{u}$, and $\omega^1, \dots, \omega^d$ form a coframe on U , we must have $g^i = \bar{g}$ for all $i = 1, \dots, d$. Hence, on U , (i) and (ii) coincide. $\square$

Lemma 5. *Suppose (CR.1), and let $(\mathcal{D}, \varphi)$ be a virtual commodity defined on some connected open set $U \subseteq \mathcal{X}$ such that the restriction of each indifference surface of $\succsim^1, \dots, \succsim^{d+1}$ to U are connected. Define $X = \frac{\partial \varphi}{\partial t} \big|_{t=0}$. Then $(\mathcal{D}, \varphi)$ satisfies (N.1) and (N.2) for every $\succsim^1, \dots, \succsim^{d+1}$ if, and only if, for all $i = 1, \dots, d$, (i) $\mathcal{L}_X \omega^i = \lambda \omega^i$, for some smooth real-valued λ , and (ii) $X^i > 0$.*

Proof. Suppose first $(\mathcal{D}, \varphi)$ is a local virtual numeraire defined on U and let $x \in U$. As $\mathcal{D}^{(x)}$ is an interval, if $t' > t > 0$, then $\mathcal{D}_t \supseteq \mathcal{D}_{t'}$, with the opposite inclusion for negative t, t' . Now, by definition, $\mathcal{D}^{(x)}$ contains some non-degenerate closed interval $[-\varepsilon, \varepsilon]$. Thus $\bigcap_{t \in [-\varepsilon, \varepsilon]} \mathcal{D}_t = \mathcal{D}_{-\varepsilon} \cap \mathcal{D}_{\varepsilon}$ is open and contains x . By possibly restricting this set further, we may, without loss of generality, suppose $W = \mathcal{D}_{-\varepsilon} \cap \mathcal{D}_{\varepsilon}$ is a connected neighborhood of x such that the restrictions of each indifference surface of every preference $\succsim^1, \dots, \succsim^{d+1}$ to W are also connected.

Trivially by (N.2) we must have $X^i > 0$. Now, by Lemma 4, for every $t \in [-\varepsilon, \varepsilon]$, since (N.1) holds, there exist some smooth $g^t : W \rightarrow \mathbb{R}_{++}$ such that, on W , $\varphi_t^* \omega^i = g^t \omega^i$ for all $i = 1, \dots, d$. Since φ is smooth in (t, x) so too is $\varphi_t^* \omega^i$ and hence so is $g^t \omega^i$. Thus, let:

$$\lambda(x) = \left. \frac{\partial g^t(x)}{\partial t} \right|_{t=0}.$$

By definition of the Lie derivative of a 1-form (Lee 2012, Equation 12.8):

$$\mathcal{L}_X(\omega^i) = \frac{\partial}{\partial t} \varphi_t^* \omega^i = \frac{\partial}{\partial t} g^t \omega^i = \lambda \omega^i$$

for all $i = 1, \dots, d$. Since (i) and (ii) are local properties and $x \in U$ was arbitrary, we have shown (N.1) and (N.2) imply (i) and (ii).

Conversely, suppose X satisfies (i) and (ii) on U . Then writing $a = \bar{u}_i|_U$, we have:

$$\begin{aligned} 0 &= \mathcal{L}_X(a du^i) \wedge (a du^i) \\ &= aX \cdot \nabla a du^i \wedge du^i + a \mathcal{L}_X du^i \wedge (a du^i) \\ &= (0) + a^2 d(X \cdot \nabla u^i) \wedge du^i. \end{aligned}$$

Now, since $a > 0$ by (CR.1), dividing off yields:

$$0 = d(X \cdot \nabla u^i) \wedge du^i.$$

Thus $d(X \cdot \nabla u^i)$ is collinear with du^i . Since it is also exact, and the level sets of u^i connected, by an analogous argument to the proof of Lemma 3, there exists some smooth, real-valued function $f^i : u^i(U) \rightarrow \mathbb{R}$ such that $X \cdot \nabla u^i = f^i \circ u^i$. By hypothesis, $X \cdot \nabla u^i = du^i(X) = X^i > 0$, hence $f^i > 0$. Thus the function $F^i : u^i(U) \rightarrow \mathbb{R}$ given by:

$$F^i(q) = \int_{\bar{q}}^q \frac{1}{f^i(\xi)} d\xi,$$

for some $\bar{q} \in u^i(U)$, is well-defined and strictly increasing. Then:

$$\begin{aligned} X \cdot \nabla F^i(u^i) &= \sum_j X^j F^{i'}(u^i) \frac{\partial}{\partial x^j} u^i \\ &= \frac{1}{f^i(u^i)} \sum_j X^j \frac{\partial}{\partial x^j} u^i \\ &= \frac{1}{f^i(u^i)} X \cdot \nabla u^i \\ &= 1. \end{aligned}$$

Thus $F^i \circ u^i$ is both a strictly increasing transformation of u^i (and hence represents $\succsim^i$ on U), and has constant X -directional derivative equal to unity, as desired. Let $(\mathcal{D}, \varphi)$ then denote the maximal flow on U associated with X ; it follows that $F^i \circ u^i|_U$ has $(\mathcal{D}, \varphi)$ as a utility unit, and hence $(\mathcal{D}, \varphi)$ is a virtual numeraire for $\succsim^i$, for all $i = 1, \dots, d$. Finally, since:

$$\mathcal{L}_X(d\bar{u}) = \sum_i \lambda \omega^i = \lambda d\bar{u},$$

an analogous argument shows $(\mathcal{D}, \varphi)$ is also a virtual numeraire for $\succsim^{d+1}$, completing the proof. $\square$

Thus locally, the satisfaction of (N.1) is equivalent to a diffeomorphism acting on the d 1-forms $\omega^1, \dots, \omega^d$ by a common, multiplicative scaling. We now expand our space to incorporate this added degree of freedom explicitly. Given $x_0 \in \mathcal{X}$ and $U, V \subseteq \mathcal{X}$ connected open sets with $x_0 \in U$, we define the **lifted** 1-forms θ^i on $\mathbb{R}_{++} \times U$ and $\mathbb{R}_{++} \times V$ via:

$$\theta^i|_{(h,u)} = h \omega^i|_u.$$

The following lemma shows that a diffeomorphism ϕ fixes every ω^i , modulo a common scaling, if and only if there exists a ‘lifted’ diffeomorphism Φ which fixes each θ^i .

Lemma 6. *Let $U, V \subseteq \mathcal{X}$ be connected open sets. Then there exists a diffeomorphism $\phi : U \rightarrow V$ such that, for all $i = 1, \dots, d$, $\phi^* \omega^i = g \omega^i$ for some smooth $g : U \rightarrow \mathbb{R}_{++}$ if, and only if, there exists a diffeomorphism $\Phi : \mathbb{R}_{++} \times U \rightarrow \mathbb{R}_{++} \times V$ such that, for all $i = 1, \dots, d$, we have $\Phi^* \theta^i = \theta^i$.*

Proof. Suppose first there exists a diffeomorphism $\Phi : \mathbb{R}_{++} \times U \rightarrow \mathbb{R}_{++} \times V$ satisfying $\Phi^* \theta^i = \theta^i$ for $i = 1, \dots, d$. Write $\Phi = (\tilde{h}(h, u), \tilde{\phi}(h, u))$. By direct computation:

$$\Phi^* \theta^i = \tilde{h}(h, u) \tilde{\phi}^* \omega^i,$$

hence:

$$\tilde{\phi}^* \omega^i = \frac{h}{\tilde{h}} \omega^i. \quad (19)$$

Now on the left-hand side of (19),

$$\tilde{\phi}^* \omega^i = \bar{u}_i(\tilde{\phi}(h, u)) \left[\sum_{j=1}^d \frac{\partial \tilde{\phi}^i}{\partial u^j} du^j + \frac{\partial \tilde{\phi}^i}{\partial h} dh \right].$$

But on the right-hand side of (19), since each ω^i has a 0 coefficient on dh , the h -partial derivative of $\tilde{\phi}$ must uniformly vanish:

$$\frac{\partial \tilde{\phi}}{\partial h} = 0.$$

Since U is connected, this implies $\tilde{\phi}$ is a function only of the utility variables $u = (u^1, \dots, u^d)$. Thus the entire left-hand side of (19) is a function solely of the u variables, and thus so too is the right. This in turn implies $h/\tilde{h}(h, u)$ is a function $g(u)$, i.e. that $h/g(u) = \tilde{h}(h, u)$, and hence the map $\Phi(h, u) = (h/g(u), \tilde{\phi}(u))$. In particular, since h and $\tilde{h}(h, u)$ are both strictly positive, so too is g , and $\tilde{\phi}^* \omega^i = g \omega^i$. Finally, in light of the above, it follows $\tilde{\phi}$ must map $U \rightarrow V$ diffeomorphically, as Φ is a diffeomorphism itself.

Conversely, suppose $\phi : U \rightarrow V$ is a diffeomorphism such that $\phi^* \omega^i = g \omega^i$ for all $i = 1, \dots, d$. Define:

$$\Phi(h, u) = \left(\frac{h}{g(u)}, \phi(u) \right).$$

Then:

$$\begin{aligned} \Phi^* \theta^i &= \frac{h}{g(u)} \phi^* \omega^i \\ &= h \frac{g}{g} \omega^i = h \omega^i = \theta^i \end{aligned}$$

as desired. □

Lemma 7. Suppose (CR.1) and (CR.2) hold, and $d \geq 3$. Then for any connected, open $U, V \subseteq \mathcal{X}$, and any $h_0 \in \mathbb{H}$ such that $h_0|_U > 0$, a diffeomorphism $\phi : U \rightarrow V$ satisfies $\phi^* \omega^i = g \omega^i$ for all $i = 1, \dots, d$ and some smooth $g : U \rightarrow \mathbb{R}_{++}$ if, and only if, $\phi^* h_0 \omega^i = h_0 \omega^i$ for all $i = 1, \dots, d$.

Proof. We begin by considering the lifted coframe elements $\theta^i = h\omega^i$. By direct computation:

$$\begin{aligned}
d\theta^i &= dh\omega^i \\
&= dh \wedge \omega^i + h d\omega^i \\
&= \frac{dh}{h} \wedge \theta^i + h \sum_{j=1}^d \frac{\bar{u}_{ij}}{\bar{u}_i \bar{u}_j} \omega^j \wedge \omega^i \\
&= \frac{dh}{h} \wedge \theta^i + \frac{1}{h} \sum_{j=1}^d \frac{\bar{u}_{ij}}{\bar{u}_i \bar{u}_j} \theta^j \wedge \theta^i.
\end{aligned} \tag{20}$$

Consider the restriction of these forms to the graph of some function $\hat{h} : U \rightarrow \mathbb{R}_{++}$. Once $\hat{h}$ is specified, since $\theta^1, \dots, \theta^d$ form a coframing of U , we have:

$$\frac{d\hat{h}}{\hat{h}} = z_1\theta^1 + \dots + z_d\theta^d,$$

for some unknown smooth, real-valued functions $z_1, \dots, z_d$ defined on U , thus (20) becomes:

$$d\theta^i = \sum_{j \neq i} \left[z_j + \frac{1}{\hat{h}} \frac{\bar{u}_{ij}}{\bar{u}_i \bar{u}_j} \right] \theta^j \wedge \theta^i.$$

Now, suppose $\phi : U \rightarrow V$ is a diffeomorphism satisfying $\phi^*\omega^i = g\omega^i$ for all $i = 1, \dots, d$, for some positive-valued smooth function g . By [Lemma 6](#), the lifted diffeomorphism $\Phi : \mathbb{R}_{++} \times U \rightarrow \mathbb{R}_{++} \times V$, defined by $\Phi(h, u) = (h/g(u), \phi(u))$, satisfies $\Phi^*\theta^i = \theta^i$ for all $i = 1, \dots, d$. Thus on the graph of $\hat{h}$:

$$\begin{aligned}
\Phi^*(d\theta^i) &= d\Phi^*\theta^i \\
&= d\theta^i \\
&= \sum_{j \neq i} \left[z_j + \frac{1}{\hat{h}} \frac{\bar{u}_{ij}}{\bar{u}_i \bar{u}_j} \right] \theta^j \wedge \theta^i.
\end{aligned}$$

But once again, since (i) $\Phi^*\theta^i = \theta^i$ for all i , and (ii) $\Phi(\hat{h}(u), u) = (\hat{h}(u)/g(u), \phi(u))$, we obtain:

$$\tilde{z}_j|_{\phi(u)} + \frac{g(u)}{\hat{h}(u)} \cdot \frac{\bar{u}_{ij}}{\bar{u}_i \bar{u}_j} \Big|_{\phi(u)} = z_j|_u + \frac{1}{\hat{h}(u)} \frac{\bar{u}_{ij}}{\bar{u}_i \bar{u}_j} \Big|_u \tag{21}$$

for all $i \neq j$, where $\tilde{z}$ denotes the analogous z functions obtained for the lifted forms on $\mathbb{R}_{++} \times V$. As such, let i, j, k be distinct. Then for every symmetry ϕ , we obtain:

$$\left[\frac{1}{h} \frac{\bar{u}_{ij}}{\bar{u}_i \bar{u}_j} - \frac{1}{h} \frac{\bar{u}_{kj}}{\bar{u}_k \bar{u}_j} \right]_{\Phi(\hat{h}(u), u)} = \left[\frac{1}{h} \frac{\bar{u}_{ij}}{\bar{u}_i \bar{u}_j} - \frac{1}{h} \frac{\bar{u}_{kj}}{\bar{u}_k \bar{u}_j} \right]_{(\hat{h}(u), u)} \tag{22}$$

by subtracting (21) for indices i and j , from the analogous equation for indices k and j . By hypothesis, for some choice of distinct i, j, k , both sides of (22) are strictly positive for all $u \in U$.⁶⁵ Define, now, for these indices, the functions $T : \mathbb{R}_{++} \times U \rightarrow \mathbb{R}_{++}$ and $h_0 : U \rightarrow \mathbb{R}_{++}$ via:

$$T(h, u) = \frac{1}{h} \left[\frac{\bar{u}_{ij}}{\bar{u}_i \bar{u}_j} - \frac{\bar{u}_{kj}}{\bar{u}_k \bar{u}_j} \right],$$

and:

$$h_0(u) = \left[\frac{\bar{u}_{ij}}{\bar{u}_i \bar{u}_j} - \frac{\bar{u}_{kj}}{\bar{u}_k \bar{u}_j} \right].$$

Clearly $h_0 \in \mathbb{H}$ and, by construction:

$$1 = T(h_0(u), u). \quad (23)$$

This implies that, for any ϕ and smooth, positive-valued function g such that $\phi^* \omega^i = g \omega^i$ for all $i = 1, \dots, d$, we have:

$$\begin{aligned} 1 &= \frac{(T \circ \Phi)(h_0(u), u)}{T(h_0(u), u)} \\ &= \frac{(T \circ \Phi)(h_0(u), u)}{1} \\ &= \frac{(h_0 \circ \phi)(u) g(u)}{h_0(u)}, \end{aligned} \quad (24)$$

where the first equality follows from dividing the left by the right-hand side of (22), the second from (23), and the final by definition of Φ . Consider now the modified coframe $h_0 \omega^i$, for $i = 1, \dots, d$. Under ϕ we have:

$$\phi^* h_0 \omega^i = (h_0 \circ \phi) g \omega^i = \left[\frac{(h_0 \circ \phi) g}{h_0} \right] h_0 \omega^i.$$

By (24), the square bracketed term is the constant function 1, hence $\phi^* h_0 \omega^i = h_0 \omega^i$ for all $i = 1, \dots, d$. Since the converse is trivial, the claim follows. $\square$

When $d = 2$, we define $\mathbb{H}$ to be the singleton set consisting of the function:

$$h_0(u) = \sqrt{\left| \frac{1}{\bar{u}_1 \bar{u}_2} \partial_{12} \ln \left(\frac{\bar{u}_1}{\bar{u}_2} \right) \right|}.$$

In such cases, (CR.2) requires the sign of this function to be locally constant.

⁶⁵This is possible as $d \geq 3$.

Lemma 8. Suppose (CR.1) and let $d = 2$. Then for any connected, open $U, V \subseteq \mathcal{X}$ such that for the unique $h_0 \in \mathbb{H}$, we have $h_0|_U > 0$, a diffeomorphism $\phi : U \rightarrow V$ satisfies $\phi^* \omega^i = g \omega^i$ for all $i = 1, 2$ and some smooth $g : U \rightarrow \mathbb{R}_{++}$ if, and only if, $\phi^* h_0 \omega^i = h_0 \omega^i$ for all $i = 1, 2$.

Proof. We once again begin by considering the lifted 1-forms θ^1 and θ^2 . By direct computation:

$$\begin{aligned} d \begin{bmatrix} \theta^1 \\ \theta^2 \end{bmatrix} &= d \begin{bmatrix} h \omega^1 \\ h \omega^2 \end{bmatrix} \\ &= \begin{bmatrix} \frac{dh}{h} & 0 \\ 0 & \frac{dh}{h} \end{bmatrix} \wedge \begin{bmatrix} \theta^1 \\ \theta^2 \end{bmatrix} + \begin{bmatrix} -\frac{1}{h} \frac{\bar{u}_{12}}{\bar{u}_1 \bar{u}_2} \\ \frac{1}{h} \frac{\bar{u}_{12}}{\bar{u}_1 \bar{u}_2} \end{bmatrix} \theta^1 \wedge \theta^2. \end{aligned}$$

Now, let $\kappa = \frac{dh}{h} + \frac{1}{h} \frac{\bar{u}_{12}}{\bar{u}_1 \bar{u}_2} \theta^1 + \frac{1}{h} \frac{\bar{u}_{12}}{\bar{u}_1 \bar{u}_2} \theta^2$. Then $d\theta^i = \kappa \wedge \theta^i$. Differentiating this, we obtain that for $i = 1, 2$:

$$\begin{aligned} 0 &= d\kappa \wedge \theta^i - \kappa \wedge d\theta^i \\ &= d\kappa \wedge \theta^i - \kappa \wedge \kappa \wedge \theta^i \\ &= d\kappa \wedge \theta^i. \end{aligned}$$

As $U \subseteq \mathcal{X}$ which is two-dimensional, this implies $d\kappa = c \theta^1 \wedge \theta^2$ for some smooth function c .

Now, suppose ϕ satisfies $\phi^* \omega^i = g \omega^i$. By Lemma 6, the map $\Phi(h, u) = (h/g(u), \phi(u))$ satisfies $\Phi^* \theta^i = \theta^i$. Then:

$$\begin{aligned} \Phi^* d\theta^i &= d\Phi^* \theta^i \\ &= d\theta^i \\ &= \kappa \wedge \theta^i. \end{aligned}$$

As pullbacks commute with wedge products, we obtain that in addition, $\Phi^* \kappa = \kappa$. From this fact, and our observation that $d\kappa = c \theta^1 \wedge \theta^2$, we conclude that $c \circ \Phi(h, u) = c(h, u)$.

Now, by direct computation, writing T for $\frac{\bar{u}_{12}}{\bar{u}_1 \bar{u}_2}$, we have:

$$\begin{aligned} d\kappa &= d \left[\frac{dh}{h} + T \omega^1 + T \omega^2 \right] \\ &= dT \wedge \omega^1 + T d\omega^1 + dT \wedge \omega^2 + T d\omega^2 \\ &= \frac{T_2}{\bar{u}_2} \omega^2 \wedge \omega^1 + \frac{T_1}{\bar{u}_1} \omega^1 \wedge \omega^2 \\ &= \left(\frac{T_1}{\bar{u}_1} - \frac{T_2}{\bar{u}_2} \right) \omega^1 \wedge \omega^2. \end{aligned} \tag{25}$$

Now, note that $T = \frac{1}{\bar{u}_1} \partial_1 \ln \bar{u}_2 = \frac{1}{\bar{u}_2} \partial_2 \ln \bar{u}_1$, hence:

$$\frac{T_1}{\bar{u}_1} = -T^2 + \frac{1}{\bar{u}_1 \bar{u}_2} \partial_{12} \ln \bar{u}_1 \quad (26)$$

and

$$\frac{T_2}{\bar{u}_2} = -T^2 + \frac{1}{\bar{u}_1 \bar{u}_2} \partial_{12} \ln \bar{u}_2. \quad (27)$$

In particular, from (25), (26), and (27) we have:

$$\begin{aligned} c(h, u) &= \frac{1}{h^2} \left[\frac{T_1}{\bar{u}_1} - \frac{T_2}{\bar{u}_2} \right] \\ &= \frac{1}{h^2} \frac{1}{\bar{u}_1 \bar{u}_2} \partial_{12} \ln \left(\frac{\bar{u}_1}{\bar{u}_2} \right). \end{aligned}$$

Since $h_0|_U > 0$, we may suppose $c > 0$ for all h and all $u \in U$ and we obtain $c(h_0(u), u) = 1$ for all $u \in U$.⁶⁶ Now, as $c \circ \Phi(h, u) = c(h, u)$:

$$\begin{aligned} 1 &= c \circ \Phi(h_0(u), u) \\ &= \left[\frac{(h_0 \circ \phi)(u) g(u)}{h_0(u)} \right]^2 \end{aligned}$$

and hence since h_0 and g are positive,

$$\frac{(h_0 \circ \phi)(u) g(u)}{h_0(u)} = 1.$$

But then:

$$\phi^* h_0 \omega^i = (h_0 \circ \phi) g \omega^i = \frac{(h_0 \circ \phi) g}{h_0} h_0 \omega^i = h_0 \omega^i,$$

as desired. □

H.5 Proof of Proposition 3

Proof. Suppose first $(\mathcal{D}, \varphi)$ is a local virtual numeraire defined on some convex open neighborhood U of x_0 . As every $\mathcal{D}^{(x)}$ is an interval, if $t' > t > 0$, then $\mathcal{D}_t \supseteq \mathcal{D}_{t'}$, with the opposite inclusion for negative t, t' . Now, by definition, $\mathcal{D}^{(x_0)}$ contains some non-degenerate closed interval $[-\varepsilon, \varepsilon]$. Thus $\bigcap_{t \in [-\varepsilon, \varepsilon]} \mathcal{D}_t = \mathcal{D}_{-\varepsilon} \cap \mathcal{D}_{\varepsilon}$ is open and contains x_0 . By possibly restricting this set further, we may, without loss of generality, suppose

⁶⁶That is, we may consider both the apparent positive and negative branches together, by simply interchanging θ^1 and θ^2 , i.e. $d\kappa = -c\theta^2 \wedge \theta^1$. See [Gardner \(1989\)](#) p. 65 for more discussion.

$W = \mathcal{D}_{-\varepsilon} \cap \mathcal{D}_{\varepsilon}$ is connected, and that the restriction of each indifference surface of every preference $\succsim^1, \dots, \succsim^{d+1}$ to it are connected. Then by [Lemma 4](#), for every $t \in [-\varepsilon, \varepsilon]$, since [\(N.1\)](#) holds, there exist some smooth $g^t : W \rightarrow \mathbb{R}_{++}$ such that, on W , $\varphi_t^* \omega^i = g^t \omega^i$ for all $i = 1, \dots, d$, where $\varphi_t^* \omega^i$ denotes the pull-back of the 1-form ω^i under φ_t .

Now, by hypothesis, $\bar{h} \in \mathbb{H}$ satisfies $\bar{h}|_W > 0$. Thus by [Lemma 7](#) and [Lemma 8](#), regardless of whether $d > 2$ or $d = 2$, we have $\varphi_t^*(\bar{h} \omega^i) = \bar{h} \omega^i$ for every $t \in [-\varepsilon, \varepsilon]$. Then, letting X denote the generator of φ , $\dot{\varphi}$, by definition of the Lie derivative of a 1-form ([Lee 2012](#), Equation 12.8):

$$\mathcal{L}_X(\bar{h} \omega^i) = \lim_{t \rightarrow 0} \frac{\varphi_t^*(\bar{h} \omega^i) - \bar{h} \omega^i}{t}$$

is well-defined using only $t \in [-\varepsilon, \varepsilon]$, and equal to zero. But in utility coordinates:

$$\begin{aligned} \mathcal{L}_X(\bar{h} \omega^i) &= X \cdot \nabla(\bar{h} \bar{u}_i) du^i + \bar{h} \bar{u}_i \mathcal{L}_X(du^i) \\ &= X \cdot \nabla(\bar{h} \bar{u}_i) du^i + \bar{h} \bar{u}_i \sum_{j=1}^d X_j^i du^j. \end{aligned}$$

Since $du^1, \dots, du^d$ form a coframe, $\mathcal{L}_X(\bar{h} \omega^i) = 0$ if and only if (i) $X_j^i = 0$ for all $i \neq j$, and (ii) $X_i^i = -X \cdot \nabla \ln(\bar{h} \bar{u}_i)$. Writing these equations in matrix form is precisely [\(16\)](#). Moreover, $X^i = X \cdot \nabla u^i = du^i(X)$, for all $i = 1, \dots, d$, hence satisfaction of [\(N.2\)](#) implies $X^i > 0$.

Conversely, suppose [\(16\)](#) admits a solution X on some neighborhood U of x_0 , and that $X^i > 0$ for all $i = 1, \dots, d$, and hence [\(N.2\)](#) obtains. By restricting, we may assume the restriction each indifference surface of every preference $\succsim^1, \dots, \succsim^{d+1}$ to U are connected. We have already shown that satisfaction of [\(16\)](#) by X is equivalent to $\mathcal{L}_X(\bar{h} \omega^i) = 0$. But:

$$0 = \mathcal{L}_X(\bar{h} \omega^i) = X \cdot \nabla \bar{h} \omega^i + \bar{h} \mathcal{L}_X(\omega^i)$$

hence, as $\bar{h} > 0$:

$$\mathcal{L}_X(\omega^i) = -X \cdot \nabla \ln(\bar{h}) \omega^i,$$

and thus $\mathcal{L}_X(\omega^i)$ can be written as $\lambda \omega^i$, for all $i = 1, \dots, d$. As a consequence, by [Lemma 5](#), the maximal flow associated with X on U is a local virtual numeraire for $\succsim^1, \dots, \succsim^{d+1}$. $\square$

H.6 Proof of Theorem 6

H.6.1 Preliminaries

Let us write the first-order system (16) as:

$$X_j^i = \psi_j^i(X, u) = \begin{cases} X \cdot \nabla K^i(u) & \text{if } i = j \\ 0 & \text{otherwise,} \end{cases} \quad (28)$$

where, as before, X_j^i denotes $\partial X^i / \partial u^j$. Define the differential operators:

$$\mathcal{D}_i = \frac{\partial}{\partial u^i} + \psi_i^i \frac{\partial}{\partial X^i},$$

for all $i = 1, \dots, d$ on $C^\infty(\mathbb{R}^d \times U)$. If X is a solution to (28), then for any smooth function $f(X(u), u)$, applying $\mathcal{D}_i$ to f corresponds to totally differentiating f with respect to u^i , then substituting ψ_j^i for X_j^i via (28).

The key integrability result for establishing the non-emptiness and dimensionality of the set of local solutions to (28) is due to [Veblen and Thomas \(1926\)](#). We give a formal statement of this result, tailored to our system, below.⁶⁷ Toward this end, for any $i \neq j$ in $1, \dots, d$, let:

$$I_j^i(X, u) = X \cdot [\nabla K_j^i(u) + K_j^i(u) \nabla K^j(u)].$$

For any family $\mathcal{F} \subset C^\infty(\mathbb{R}^d \times U)$, let $\langle \mathcal{F} \rangle_U$ denote the set of all finite, linear combinations of functions in $\mathcal{F}$, whose coefficients are smooth functions $U \rightarrow \mathbb{R}$. Then, define:

$$\mathcal{I}^0 = \langle I_j^i(X, u) : i \neq j \rangle_U,$$

and inductively, let: $\mathcal{I}^{L+1} = \langle \mathcal{I}^L \cup \{\mathcal{D}_j I : I \in \mathcal{I}^L, j = 1, \dots, d\} \rangle_U$ for all $L \geq 0$, and let $\mathcal{Z}^L = \{(X, u) \in \mathbb{R}^d \times U : I(X, u) = 0 \text{ for all } I \in \mathcal{I}^L\}$. We note that the sets $\mathcal{Z}^L$ could be equivalently obtained without taking the $C^\infty(U)$ -spans, $\langle \cdot \rangle_U$, as formation of combinations of this type leaves the common zero set unchanged.

Theorem ([Veblen and Thomas 1926](#)). *Let $(X_0, u_0) \in \mathbb{R}^d \times U$ and suppose that, in some neighborhood, of (X_0, u_0) , there exist $L^* \geq 0$ and some tuple of functions $\hat{I} = (\hat{I}^1, \dots, \hat{I}^p) \subseteq \mathcal{I}^{L^*}$ such that:*

⁶⁷The original proof in [Veblen and Thomas \(1926\)](#) or [Eisenhart \(1927\)](#) is very terse, and only informally states several needed regularity conditions. We provide a more formal statement here.

(i) The matrix:

$$\left[\frac{\partial \hat{I}^i}{\partial X^j} \right]_{1 \leq i \leq p, 1 \leq j \leq d}$$

has constant rank p in a neighborhood of (X_0, u_0) .

(ii) $\mathcal{Z}^{L^*} = \mathcal{Z}^{L^*+1}$, and locally about (X_0, u_0) , and $\mathcal{Z}^{L^*}$ is precisely the common zero set of $\hat{I}^1, \dots, \hat{I}^p$.

(iii) $(X_0, u_0) \in \mathcal{Z}^{L^*}$.

Then there exists a neighborhood $\mathcal{N}_0$ of u_0 and a unique solution X to (28) on $\mathcal{N}_0$, with $X(u_0) = X_0$. Moreover, the family of local solutions is indexed by the $\tilde{X}_0$ such that $(\tilde{X}_0, u_0) \in \mathcal{Z}^{L^*}$ and hence depends smoothly on $d - p$ constants.

Finally, we supply a book-keeping lemma which relates the matrices $\mathcal{M}$ to the functions in each $\mathcal{I}^L$.

Lemma 9. For any tuple $\alpha = (i, j_0, \dots, j_L)$, $i \neq j_0$, let:

$$R_\alpha(X, u) = X \cdot r_\alpha(u) \quad \text{and} \quad r_\alpha(u) = \nabla Q_{j_0 \dots j_L}^i + Q_{j_0 \dots j_L}^i \sum_{l=0}^L \nabla K^{j_l}.$$

Then for any $0 \leq \bar{L}$,

$$\mathcal{I}^{\bar{L}} = \langle R_\alpha : \alpha = (i, j_0, \dots, j_L), i \neq j_0, L \leq \bar{L} \rangle_U, \quad (29)$$

and hence:

$$\mathcal{Z}^{\bar{L}} = \{(X, u) \in \mathbb{R}^d \times U : \mathcal{M}_{\bar{L}}(u)X = 0\}.$$

Proof. Fix $\alpha = (i, j_0, \dots, j_L)$, $i \neq j_0$. Then, for any $k = 1, \dots, d$, by direct calculation:

$$\begin{aligned} \mathcal{D}_k \left(X \cdot \left[\nabla Q_{j_0 \dots j_L}^i + Q_{j_0 \dots j_L}^i \sum_{l=0}^L \nabla K^{j_l} \right] \right) &= \sum_{a=1}^d X^a \left[K_a^k \frac{\partial}{\partial u^k} Q_{j_0 \dots j_L}^i + K_a^k Q_{j_0 \dots j_L}^i \sum_{l=0}^L K_k^{j_l} + \dots \right. \\ &\quad \left. \dots + \frac{\partial^2}{\partial u^a \partial u^k} Q_{j_0 \dots j_L}^i + \left(\frac{\partial}{\partial u^k} Q_{j_0 \dots j_L}^i \right) \sum_{l=0}^L K_a^{j_l} + Q_{j_0 \dots j_L}^i \sum_{l=0}^L K_{ak}^{j_l} \right] \\ &= \sum_{a=1}^d X^a \left[K_a^k Q_{j_0 \dots j_L k}^i + \frac{\partial^2}{\partial u^a \partial u^k} Q_{j_0 \dots j_L}^i + \dots \right. \\ &\quad \left. \dots + \left(Q_{j_0 \dots j_L k}^i - Q_{j_0 \dots j_L}^i \sum_{l=0}^L K_k^{j_l} \right) \sum_{l=0}^L K_a^{j_l} + Q_{j_0 \dots j_L}^i \sum_{l=0}^L K_{ka}^{j_l} \right] \end{aligned}$$

Expanding the cross-derivative of $Q_{j_0 \dots j_L}^i$ using the definition of $Q_{j_0 \dots j_L}^i$, and canceling terms, yields:

$$\begin{aligned}
\cdots &= \sum_{a=1}^d X^a \left[Q_{j_0 \dots j_L}^i \left(K_a^{j_0} + \cdots + K_a^{j_L} + K_a^k \right) + \frac{\partial}{\partial u^a} Q_{j_0 \dots j_L}^i + \cdots \right. \\
&\quad \left. \cdots - \left(\frac{\partial}{\partial u^a} Q_{j_0 \dots j_L}^i + Q_{j_0 \dots j_L}^i \left(\sum_{l=0}^L K_a^{j_l} \right) \right) \left(\sum_{l=0}^L K_k^{j_l} \right) \right] \\
&= X \cdot \left[\nabla Q_{j_0 \dots j_L}^i + Q_{j_0 \dots j_L}^i (\nabla K^{j_0} + \cdots + \nabla K^{j_L} + \nabla K^k) - \cdots \right. \\
&\quad \left. \cdots - \left(\sum_{l=0}^L K_k^{j_l} \right) \left(\nabla Q_{j_0 \dots j_L}^i + Q_{j_0 \dots j_L}^i \sum_{l=0}^L \nabla K^{j_l} \right) \right].
\end{aligned}$$

The dot product of X with first summand in the square brackets is simply $R_{\alpha k}$, where $\alpha k = (i, j_0, \dots, j_L, k)$. Similarly, the dot product of X with the second summand is $(\sum_{l=0}^L K_k^{j_l})$ times R_α . Thus:

$$R_{\alpha k}(X, u) = \mathcal{D}_k R_\alpha + \left(\sum_{l=0}^L K_k^{j_l} \right) R_\alpha.$$

For $\bar{L} = 0$, the $R_{i j_0}$ are precisely the $I_{j_0}^i$ hence (29) holds trivially. Suppose then that (29) holds for some $\bar{L} \geq 0$. Then the above calculation (and Leibniz's rule) shows that $\mathcal{D}_k$ of any element of the right-hand side of (29) belongs to the span of the functions $R_{\alpha'}$ where α' is of length $\leq \bar{L} + 1$. Conversely, the above calculation shows that each new generator $R_{\alpha k}$ belongs to the span of $\mathcal{I}^L$ and its derivatives under the operators $\mathcal{D}_k$. Thus (29) holds for indices of length $\bar{L} + 1$, and the claim follows by induction. Finally, note the rows of $\mathcal{M}_{\bar{L}}(u)$ are precisely the transposes of the coefficients r_α , hence the final claim is immediate. $\square$

H.6.2 Proof of Theorem 6

Proof. ($\Leftarrow$): Suppose that $\ker \mathcal{M}_{L^*}(u_0) \cap \mathbb{R}_{++}^d \neq \emptyset$, and suppose X_0 is any arbitrary element of this intersection. Let $p = \text{rank } \mathcal{M}_{L^*}(u_0)$. Since $X_0 \neq 0$ belongs to this kernel, $p < d$. Now, by definition of L^* ,

$$\text{rank } \mathcal{M}_{L^*}(u_0) = \text{rank } \mathcal{M}_{L^*+1}(u_0),$$

and by (CR.3), after restricting U to some sufficiently small neighborhood of u_0 , both matrices have constant rank p . We will henceforth suppose U has been replaced with

any such restriction. As the rows of $\mathcal{M}_{L^*}$ are included in those of $\mathcal{M}_{L^*+1}$, we have for every $u \in U$:

$$\ker \mathcal{M}_{L^*}(u) = \ker \mathcal{M}_{L^*+1}(u).$$

Choose p rows, $r^1(u)^\top, \dots, r^p(u)^\top$ of $\mathcal{M}_{L^*}(u_0)$ that are linearly independent at u_0 (if $p = 0$ we instead choose the empty tuple). By possibly restricting U further, we may suppose that $r^1(u)^\top, \dots, r^p(u)^\top$ are linearly independent at every $u \in U$; we suppose this is true. Define:

$$\hat{I}^a(X, u) = X \cdot r^a(u), \quad a = 1, \dots, p.$$

By [Lemma 9](#), $\hat{I}^a \in \mathcal{I}^{L^*}$ for every $a = 1, \dots, p$. Moreover, the Jacobian with respect to the X variables of the vector valued function $(\hat{I}^1, \dots, \hat{I}^p)$ is simply the matrix whose rows are $r^1(u)^\top, \dots, r^p(u)^\top$, and hence has constant rank p . Moreover, because $\mathcal{M}_{L^*}$ also has rank p everywhere on U , these rows span the row-space of $\mathcal{M}_{L^*}(u)$ for all $u \in U$. Thus, again by [Lemma 9](#),

$$\mathcal{Z}^{L^*} = \{(X, u) \in \mathbb{R}^d \times U : \mathcal{M}_{L^*}(u)X = 0\},$$

and $\mathcal{Z}^{L^*} = \mathcal{Z}^{L^*+1}$. Thus by the theorem of [Veblen and Thomas \(1926\)](#), every initial condition $\tilde{X}_0 \in \ker \mathcal{M}_{L^*} \cap \mathbb{R}_{++}^d$ determines a unique local solution about u_0 . By [Proposition 3](#), this solution then corresponds to a local, common virtual numeraire for the subprofile $(\succsim^1, \dots, \succsim^{d+1})$.

($\implies$) : Suppose instead that some common, local virtual numeraire φ exists. Then by [Proposition 3](#), $X = \dot{\varphi}$ is a positive, local solution to (28) on some neighborhood of u_0 .

For any $i \neq j$, we have:

$$\begin{aligned} I_j^i(X(u), u) &= (\mathcal{D}_j \psi_i)(X(u), u) \\ &= \frac{\partial}{\partial u^j} \left(\frac{\partial X^i}{\partial u^i} \right) \\ &= \frac{\partial}{\partial u^i} \left(\frac{\partial X^i}{\partial u^j} \right) = 0. \end{aligned}$$

Hence every element of $\mathcal{I}^0$ vanishes along the graph of X . Now, suppose that for every $0 \leq L \leq \bar{L}$, every element of $\mathcal{I}^L$ vanishes on the graph of X . For any $I \in \mathcal{I}^L$ and $j = 1, \dots, d$,

$$(\mathcal{D}_j I)(X(u), u) = \frac{\partial}{\partial u^j} I(X(u), u),$$

as $\mathcal{D}_j$ corresponds to simply total differentiation of I along the graph of X , and I is constant on this graph. Because $\mathcal{I}^{\bar{L}+1}$ is the U -span of $\mathcal{I}^{\bar{L}}$ and these derivatives, every element of $\mathcal{I}^{\bar{L}+1}$ must also vanish on the graph of X . By induction, we obtain that $(X(u), u) \in \mathcal{Z}^L$ for all $L \geq 0$. Now, by [Lemma 9](#), for all $L \geq 0$ we also have $\mathcal{M}_L(u)X(u) = 0$. In particular, $X(u_0) \in \ker \mathcal{M}_L(u_0) \cap \mathbb{R}_{++}^d$ as desired. $\square$

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