Preference Regression*

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Abstract

This paper investigates the problem of model selection and testing in decision theory. We consider a consumption space equipped with an endogenous notion of abstract 'numeraire,' and characterize those preferences for which the quantity of numeraire needed to compensate an agent between a pair of alternatives provides a consistent, cardinal measure of the magnitude of preference. This framework includes all quasilinear or homothetic preferences on classical consumption spaces, stationary preferences over dated rewards, von Neumann-Morgenstern preferences on lottery spaces, and a wide range of preferences over monetary acts, including those represented by subjective expected utility, Choquet expected utility, maxmin expected utility, variational, and dual-self functionals. For data consisting of observed or experimentally elicited compensation differences, we show a simple least-squares methodology provides a systematic means of estimating the 'best-fit' preferences for an inconsistent data set from a given model, and allows for meaningful comparisons of goodness of fit across models. In the presence of cross-sectional data, our approach allows for nonparametric statistical testing of rationalizability at the population level.

1 Introduction

Decision theory features a wealth of models, often aiming to describe similar phenomena. Comparatively, the theory of inconsistency measurement and model selection has been less well developed.

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A robust, generally applicable approach to model selection serves not only to discipline the creation of new models, but also to refine our understanding of those which already exist. However, much of the literature to date on these topics has been model-specific, and hence unable to make cross-model comparisons.

In this paper we establish a general, least squares methodology for the related problems of inconsistency quantification and model selection in decision theory. We show that for a broad class of preferences, a natural form of experiment yields data for which these problems reduce to solving a least squares program with convex constraint. For such data, testing consistency with respect to a particular model generally amounts to testing whether or not a vector of observations belongs to a fixed, polyhedral set of rationalizable vectors. Model selection may be conducted by determining which convex set in a particular family lies closest to the data vector. For example, to compare whether Choquet or maxmin expected utility better describes the ambiguity attitudes in some data, it suffices to compute the projection of the data vector onto the two polyhedral sets representing the models, and compare the resulting distances. When data are repeatedly sampled from a fixed population, we leverage this structure to provide an explicit, nonparametric statistical test of aggregate rationalizability.

Relative to the existing literature, our approach provides three noteworthy advantages. Firstly, while many measures of consistency provide a numeric indication of the goodness of fit for a particular model, they do not speak to *which* particular preferences from the class of interest are the most consistent.[1] While our regression framework does not, in general, identify a single preference as the 'best fit' from a given model, it does identify a set of such preferences. This may be interpreted as identifying the particular *structure* of those preferences which best reflect the (finite) data.[2] Secondly, many existing measures of inconsistency are model-specific, and hence unable to address questions about comparative goodness of fit across models. In contrast, our methods are general, allowing for their simultaneous application to multiple models and hence to questions of model selection. Finally, while we provide explicit statistical tests of rationalizability when presented with stochastic data, many measures of inconsistency do not. This allows our

[1] For example, while the popular Afriat and Varian efficiency indices provide a means of quantifying the degree of failure of the strong axiom of revealed preference, they provide no means of ascertaining *which* rational preference(s) best approximate the data.

[2] The set-valued identification is a consequence of our assumption of finite experiments; any two preferences in the best fit set will necessarily be indistinguishable by the experiment at hand.

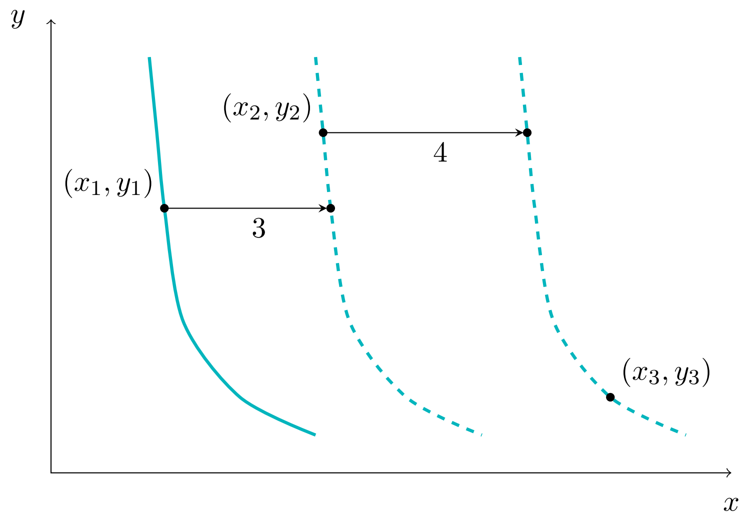


Figure 1: When preferences are representable by a quasilinear utility of the form $u(x, y) = v(y) + x$, indifference curves are horizontal translates of one another. Geometrically, translating any point on the (x_1, y_1) indifference curve 3 units to the right makes it fall precisely on the (x_2, y_2) indifference curve. If one translates the result again by 4, the point must then lie on the (x_3, y_3) curve.

approach a means of objectively quantifying the degree to which the data fit a given model that is unavailable to many other methods.

Example 1. Suppose one wishes to investigate how well-approximated a particular individual's preference over bundles of two commodities are by a continuous quasilinear utility function of the form:[3]

$$U(x, y) = v(y) + x.$$

Preferences representable by such a utility are invariant under receiving more of the numeraire commodity, x. For example, $(3, 5) \succsim (4, 2)$ if and only if for all $\alpha \geq 0$, $(3 + \alpha, 5) \succsim (4 + \alpha, 2)$. In particular, this implies the indifference curves of any such preference are horizontal translates of one another.

Consider the three bundles $(x_1, y_1), (x_2, y_2)$, and (x_3, y_3) indicated in Figure 1. Suppose the subject is presented with all three possible pairs of bundles from this collection and one observes, for each pair of bundles, both (i) which bundle is preferred, and (ii) what quantity of numeraire,

[3] Our use of the phrase 'the individual's preference' here is meant informally; in particular we do not assume, a priori, that the individual's choice behavior is consistent with the maximization of any complete and transitive binary relation.

in conjunction with receiving the less-preferred bundle, would make the subject indifferent relative to receiving the more-preferred bundle. It is observed first that the subject is indifferent between $(x_1 + 3, y_1)$ and (x_2, y_2), and similarly between $(x_2 + 4, y_2)$ and (x_3, y_3). If the subject did possess a quasilinear preference, this would imply that translating the (x_1, y_1) indifference curve 3 units to the right makes it perfectly overlap the (x_2, y_2) indifference curve, and likewise, translating the (x_2, y_2) indifference curve 4 units to the right makes it overlap the (x_3, y_3) curve. Thus, for the data to be consistent with any quasilinear preference, it is necessary that, for the third pair, the subject be indifferent between $(x_1 + 7, y_1)$ and (x_3, y_3). In fact, this 'adding-up' condition is also sufficient for the existence of a continuous, quasilinear rationalizing utility.[4]

More generally, the set of 'quasilinear-rationalizable' data sets are precisely those which satisfy the above adding-up constraint. These vectors form a linear subspace (here, a plane) in the space of all possible data vectors that could result from the experiment. Given data *inconsistent* with these hypotheses, there is a unique mean squared error minimizing choice of consistent dataset, obtained by orthogonally projecting the data onto the subspace of rationalizable vectors. The distance between the original data and its 'best fit' serves as a natural measure of goodness of fit for the hypothesis of quasilinearity.

Finally, it is straightforward to further restrict to only those vectors rationalizable by quasilinear utilities with additional structure. For example, if one wished to additionally require $v(y)$ be increasing and concave, the set of rationalizable vectors would form a polyhedral subset of the quasilinear-rationalizable plane.[5] To compute the best fit, one would simply project the data onto this subset instead, and the distance between the data and its projection provides a quantification of the goodness of fit.

2 Related Literature

The primary contribution of this paper is to decision theory. Our first main result provides necessary and sufficient conditions for a preference to admit a so-called additive-equivariant utility. The existence of such a representation guarantees the observable, but in general unstructured, compensation differences coincide with highly structured but unobservable utility differences for all pairs of alternatives. Leveraging this overlap, our Theorem 3 provides a complete characterization of the

[4] See Theorem 3.

[5] For an explicit description of this set for a general experiment, see Example 7.

empirical content of such preferences for compensation differences data. We show the set of rationalizable data sets forms a linear subspace of the space of all data, for any given experiment. This allows for natural and efficient means of computing both the additive-equivariant preferences that best fit a given collection of data, and of quantifying the inconsistency present in non-rationalizable data sets. We show that a natural extension of the money pump metric of Echenique et al. (2011) provides a convenient, economic measure of inconsistency for non-rationalizable data.[6]

While our results hold across a variety of decision theoretic models, the literature on preferences under ambiguity is a rich source of applications for our methodology. We focus on preferences over monetary acts, as in, for example Rigotti et al. (2008), Bossaerts et al. (2010), Bayer et al. (2013), Ahn et al. (2014), or Chambers et al. (2016).[7] For any data set of pairwise compensation differences, we provide characterizations of the empirical content of a variety of models of preference under ambiguity, including subjective expected utility (Anscombe and Aumann 1963), Choquet expected utility (Schmeidler 1989), maxmin expected utility (Gilboa and Schmeidler 1989), variational (Maccheroni et al. 2006), as well as dual-self and dual-self variational preferences (Chandrasekher et al. 2020).[8] Taken as a whole, our results yield not only a class of experiments capable of simultaneously differentiating between these models, but also revealed preference-like characterizations and explicit statistical tests for each.

Compensation differences involve two distinct pieces of information: which alternative in a particular pair is more preferred, and by how much. We introduce an extension of the Becker-DeGroot-Marschak mechanism capable of simultaneously and truthfully eliciting both these unknowns in experimental settings. The form of experiment considered here may then be viewed as a mixture of two standard methods of eliciting preferences over risky or uncertain prospects: having subjects make pairwise comparisons (e.g., Hey et al. 2010, Abdellaoui et al. 2011) and eliciting

[6]Our methods differ from the recent contribution of Polisson et al. (2020) both in terms of the type of data we take as primitive, and the approach taken to quantify inconsistency. As such, while, Polisson et al.'s results hold for a number of cases in which ours do as well (e.g., maxmin expected utility), we view their findings as complementary.

[7]Preferences over monetary acts may alternatively be interpreted as 'risk-neutral' preferences over a richer domain that includes both subjective and objective uncertainty, but where each monetary lottery has been replaced with its certainty equivalent.

[8]As noted in Chandrasekher et al. (2020), dual-self expected utility is a particular choice of representation of the class of invariant biseparable preferences of Ghirardato et al. (2004). As our results are ordinal in nature, all our results on empirical content and testing hold true for the underlying class of preferences, rather than resting upon a particular choice of representation.

reservation prices (e.g., Becker et al. 1964, Halevy 2007).[9]

This paper contributes to a growing recent literature on the statistical testing of various decision- and demand-theoretic models. Much of this work has focused on constructing model-specific tests, for example Kitamura and Stoye (2018), Deb et al. (2018), Fudenberg et al. (2020), Cattaneo et al. (2020), and Smeulders et al. (2021). In constrast, we put forward a general methodology that applies to a wide range of different models, over a variety of domains. To obtain critical values for our test, we make use of an implementation of the non-differentiable delta method of Fang and Santos (2019) due to Hong and Li (2020). A notable benefit of this approach is its flexibility: we provide a means of testing both individual axioms and whole models, allowing far more granular insights into not only which models may fail to be consistent with the data, but which aspects of the model are most responsible for the rejection.[10]

Finally, the use of least-squares techniques to aggregate incomplete and potentially inconsistent observations into a coherent ranking has a long history, e.g. Harville (1977), Stefani (1977). Recently, there has been renewed mathematical interest in the problem of establishing an optimal statistical ranking from an inconsistent, incomplete, and noisy dataset. We draw on a number of ideas from this literature, including the representation of data as a flow on a particular network whose structure reflects the incompleteness our observations, and the various associated regression theories for such problems, see Hirani et al. (2010), Jiang et al. (2011), Osting et al. (2013) and references therein. These ideas have already found application elsewhere in economics, including game theory, social choice, and revealed preference (resp. Candogan et al. 2011, Csató 2015, Caradonna 2020).

3 Invariant Preferences

3.1 Model

Let (X, d) be a metric space of **alternatives** that an agent has preferences over.[11] We say a jointly continuous function $\phi : \mathbb{R}_+ \times X \to X$ defines a **virtual numeraire** commodity if (i) for all $x \in X$,

[9] This stands in contrast to the 'allocation' approach of Loomes (1991), Andreoni and Miller (2002), Choi et al. (2007), Hey and Pace (2014), and Ahn et al. (2014).

[10] Our method of constructing test statistics also draws heavy from the statistical and econometric literature on shape-constrained regression, in particular Allon et al. (2007), Kuosmanen (2008), and Seijo and Sen (2011).

[11] By a preference, we mean a complete and transitive binary relation on X, which we will denote by $\succsim$. As is standard, we use $\succ$ and $\sim$ to denote the asymmetric (resp. symmetric) components.

$\phi(0, x) = x$, and (ii) for all $\alpha, \beta \in \mathbb{R}_+$ and all $x \in X$, $\phi\big(\beta, \phi(\alpha, x)\big) = \phi(\alpha + \beta, x)$. Formally, such a map ϕ defines a continuous **action** of the monoid $\mathbb{R}_+$ on X.[12] For our purposes, ϕ corresponds to a collection of transforms $X \to X$, one for each $\alpha \geq 0$, which we interpret as augmenting an alternative with some quantity of virtual numeraire. Thus for any $\alpha \geq 0$ and any $x \in X$, we interpret $\phi(\alpha, x)$ as x plus α additional units of numeraire. Property (i) ensures that the transform corresponding to adding no units numeraire does not alter any alternative; property (ii) is a path-independence condition: adding β units of numeraire to the alternative consisting of x plus α units of numeraire is the same as simply adding $\alpha + \beta$ units of numeraire to x at once.

A preference $\succsim$ on X is continuous if, for all $x \in X$, $\{x' \in X : x' \succsim x\}$ and $\{x' \in X : x \succsim x'\}$ are closed. Given a space X equipped with a virtual numeraire ϕ, we will consider those continuous preferences which satisfy the following three properties:

(N.1) **Invariance**: For all $\alpha \in \mathbb{R}_+$, $x, y \in X$:

$$x \succsim y \iff \phi(\alpha, x) \succsim \phi(\alpha, y).$$

(N.2) **Monotonicity**: For all $\alpha \in \mathbb{R}_+$, $x \in X$:

$$\phi(\alpha, x) \succsim x,$$

with indifference if and only if $\alpha = 0$.

(N.3) **Compensability**: For all $x, y \in X$,

$$x \succ y \implies \exists\, \alpha \in \mathbb{R}_+ \text{ s.t. } \phi(\alpha, y) \sim x.$$

Invariance says that adding the same quantity of numeraire to two alternatives does not affect the preference between them. It rules out numeraire-based 'wealth effects' where, when coupled with a high enough quantity of additional numeraire, an agent's preferences between two alternatives reverses. Monotonicity says the virtual numeraire commodity is a good. Compensability is a richness condition for ϕ. It ensures that any preference between two alternatives can always be offset by some quantity of numeraire. It rules out lexicographic behavior where no amount of numeraire could compensate an agent for receiving a less-preferred alternative.

[12] A monoid is a semigroup with identity; see Fuchs (2011). Formally, $\mathbb{R}_+$, equipped with the the usual notion of addition $+$, is a monoid.

If $y \succsim x$, then (N.3) guarantees there is some α such that $\phi(\alpha, x) \sim y$. We term this α the **compensation difference** from x to y. Note that by (N.2) this quantity is necessarily unique. Together, the axioms (N.1) - (N.3) will be shown to characterize those preferences which admit a representation under which the compensation difference, measured in numeraire units between any pair of alternatives, is precisely the utility difference of the alternatives.

3.2 Examples

Example 2. (Quasilinear Preferences): Suppose $X = \mathbb{R}^2_+$, and let $\phi\big(\alpha, (x, y)\big) = \big(\alpha + x, y\big)$. Any continuous preference $\succsim$ that admits a utility of the form $U(x, y) = v(y) + x$ clearly satisfies (N.2)-(N.3). Moreover,

$$\begin{aligned} &(x, y) \succsim (x', y') \\ \iff\ & v(y) + x \geq v(y') + x' \\ \iff\ & v(y) + (\alpha + x) \geq v(y') + (\alpha + x') \\ \iff\ & \phi\big(\alpha, (x, y)\big) \succsim \phi\big(\alpha, (x', y')\big), \end{aligned}$$

and thus $\succsim$ also obeys (N.1). Here, the compensation difference measures the quantity of the first commodity needed to offset a strict preference between two bundles.

Example 3. (Stationary Preferences for Dated Rewards): Suppose $X = \mathbb{R}_+ \times Z$ with $\phi\big(\alpha, (t, z)\big) = (\alpha + t, z)$ where, following Fishburn and Rubinstein (1982), a pair $(t, z) \in X$ represents delivery of some alternative $z \in Z$ at t units of time in the future.[13] It is natural, when prizes are goods, to instead require (N.2) to hold with the opposite relation, to reflect impatience on the part of the agent. Suppose $\succsim$ admits a utility of the form:

$$U(t, z) = \rho^t v(z),$$

where $0 < \rho < 1$ and $v : Z \to \mathbb{R}_{++}$. Then $\succsim$ satisfies (N.1) by an analogous argument to the preceding example.[14] Moreover, for any (t, z) and any $\alpha \geq 0$, $\rho^{t+\alpha} v(z) < \rho^t v(z)$. Hence the reverse analogue of (N.2) holds. Similarly, if $(t, z) \succ (t', z')$, then for some $\alpha > 0$, $U(\alpha + t, z) = U(t', z')$

[13] Ok and Masatlioglu (2007) provide a complementary interpretation of preferences over X as the commitment preferences of an agent.

[14] In this setting, (N.1) corresponds to Fishburn and Rubinstein (1982)'s (A.2), a monotonicity axiom, and (A.5), their stationarity axiom (cf. the stationarity axiom of Ok and Masatlioglu 2007). Our axiom (N.2) is closely related to Fishburn and Rubinstein's axiom (A.3); they obtain (N.3) as a consequence of a continuity axiom and the particular topological structure of the consumption space.

hence the reverse analogue of (N.3) is satisfied. In this setting, the compensation difference measures the amount of time one must postpone the delivery of the more desirable dated reward to achieve indifference.

Example 4. (Homothetic Preferences): Let $X = \mathbb{R}^L_+ \setminus \{0\}$ represent the standard demand theoretic consumption space minus the origin, and define $\phi : \mathbb{R}_+ \times X \to X$ via:

$$\phi(\alpha, x) = e^{\alpha} x.$$

Suppose a preference $\succsim$ admits a utility U that is homogeneous of degree one.[15] The relation $\succsim$ satisfies (N.2) and (N.3) if, for example, U is strictly positive on X. It necessarily also satisfies (N.1). Indeed the preference is homothetic if and only if $\succsim$ satisfies (N.1). To see this, suppose $x \succsim y$. For any $\lambda \geq 1$, $\lambda x \succsim \lambda y$ is equivalent to $e^{\ln\lambda} x \succsim e^{\ln\lambda}$ and hence $\lambda x \succsim \lambda y$ follows from (N.1) with $\alpha = \ln \lambda$. If instead $\lambda < 1$, by (N.1):

$$\begin{aligned} &\lambda x \succsim \lambda y \\ \iff &e^{\ln\frac{1}{\lambda}} \lambda x \succsim e^{\ln\frac{1}{\lambda}} \lambda y \\ \iff &x \succsim y, \end{aligned}$$

and as $x \succsim y$ by hypothesis, it again follows that $\lambda x \succsim \lambda y$. Thus (N.1), for this choice of ϕ, simply corresponds to homotheticity.[16] Here compensation differences measure the amount, holding proportions fixed, one would need to scale up the less-preferred bundle of commodities to achieve indifference.

Example 5. (Mixture Independence): Let $\tilde{X}$ be a finite set, and let $\Delta(\tilde{X})$ denote the set of lotteries supported on $\tilde{X}$. Let $\bar{x} \in \tilde{X}$ be arbitrary. Define $\phi_{\bar{x}} : \mathbb{R}_+ \times \Delta(\tilde{X}) \to \Delta(\tilde{X})$ via:

$$\phi_{\bar{x}}(\alpha, p) = e^{-\alpha} p + (1 - e^{-\alpha})\delta_{\bar{x}},$$

where $\delta_{\bar{x}}$ denotes a dirac measure or point mass centered at $\bar{x}$. This defines a virtual numeraire, as:

$$\begin{aligned} \phi_{\bar{x}}\big(\beta, \phi(\alpha, p)\big) &= e^{-\beta}\big(e^{-\alpha} p + (1 - e^{-\alpha})\delta_{\bar{x}}\big) + (1 - e^{-\beta})\delta_{\bar{x}} \\ &= e^{-(\beta+\alpha)} p + \big(1 - e^{-(\beta+\alpha)}\big)\delta_{\bar{x}} \\ &= \phi_{\bar{x}}(\beta + \alpha, p), \end{aligned}$$

[15] A function U is said to be homogeneous of degree one if, for all $\alpha > 0$, $U(\alpha x) = \alpha U(x)$.

[16] More generally, there is no added implication from considering an invariance axiom for an action of the full group $(\mathbb{R}, +)$ (or, here, the isomorphic group $(\mathbb{R}_{++}, \cdot)$) where possible, because of the 'if and only if' in (N.1). In particular, even if ϕ extends from an action of $\mathbb{R}_+$ to an action of $\mathbb{R}$, the set of invariant preferences will be the same. This is true generally, see for example, Cerreia-Vioglio et al. (2014), footnote 5.

and $\phi(0,p) = p$. Now, let $\succsim$ be any von Neumann-Morgenstern preference on $\Delta(\tilde{X})$ that ranks $\delta_{\bar{x}}$ as the unique, most-preferred lottery. Then the restriction of $\succsim$ to $\Delta(\tilde{X}) \setminus \{\delta_{\bar{x}}\}$ satisfies (N.1) to (N.3).[17] Compensation differences here measures how much one would need to mix the less-desirable lottery with the sure-thing $\delta_{\bar{x}}$, before a subject deems the resulting lottery as good as the more-desirable lottery in a pair.

Example 6. (Translation-Invariant Preferences): Let S be a finite set of states of the world, and $X = \mathbb{R}^S$ denote the space of all real-valued (monetary) acts.[18] Define $\phi : \mathbb{R}_+ \times X \to X$ via $\phi(\alpha, x) = x + \alpha \mathbb{1}_S$. A function $U : X \to \mathbb{R}$ is said to be translation-invariant if, for all $\alpha \in \mathbb{R}$,

$$U\big(x + \alpha \mathbb{1}_S\big) = U(x) + \alpha.$$

Then any preference $\succsim$ on X that admits a translation-invariant utility satisfies (N.1) to (N.3).[19] Grant and Polak (2013) interpret translation-invariance over utility acts as reflecting constant absolute ambiguity aversion. Interpreting acts as portfolios of Arrow securities, compensation differences in this setting correspond to the quantity of bonds (i.e. assets paying off the same across each possible state) an agent with risk-neutral preferences must additionally be awarded, in addition to a less preferred portfolio, to be indifferent with holding a more preferable one.

3.3 Representation

In this section, we will establish a utility representation for any preferences satisfying (N.1) to (N.3). We say that a utility $U : X \to \mathbb{R}$ is **additive-equivariant** if, for all $\alpha \in \mathbb{R}_+$, and all $x \in X$:[20]

$$U\big(\phi(\alpha, x)\big) = U(x) + \alpha.$$

[17] That $\delta_{\bar{x}}$ is the unique preference-maximal lottery is required to ensure (N.2) and (N.3). If there were some $p' \succsim \delta_{\bar{x}}$, then (N.2) would fail for $\phi(\alpha, p')$, as adding more numeraire corresponds to increasing the mixing coefficient on $\delta_{\bar{x}}$, which must weakly decrease utility. This is also motivates the restriction of $\succsim$ to $X \setminus \{\delta_{\bar{x}}\}$. Similarly, if there were p, p' such that $p \precsim \delta_{\bar{x}} \prec p'$, there would fail to be any compensation value for the pair $\{p, p'\}$ under $\phi_{\bar{x}}$.

[18] We consider finite S to avoid measurability concerns. The example remains true if, for example, one has a measurable space (S, Σ) and X is a cone of real-valued measurable maps that contains the constant functions.

[19] By an argument analogous to that in the homothetic preferences example, even though translation invariance allows for negative α, it is still implied by (N.1).

[20] Given a monoid M, and actions ϕ_X and ϕ_Y of M on sets X and Y, a map $f : X \to Y$ is said to be **equivariant** if, for all $m \in M$ and all $x \in X$:

$$f\big(\phi_X(m, x)\big) = \phi_Y\big(m, f(x)\big).$$

Additive-equivariance corresponds to the special case where M is $\mathbb{R}_+$, Y is $\mathbb{R}$, and the action ϕ_Y is simply addition.

Additive-equivariance is a 'coordinate-free' generalization of quasilinearity, phrased purely in terms of the interaction of U and the virtual numeraire ϕ. It requires that, for every x, that the addition of α units of numeraire yields precisely α extra utility on top of the utility from x. In particular, if α is the compensation difference from x to y, then for any additive-equivariant utility, $U(y) - U(x) = \alpha$.

Theorem 1. *Suppose that ϕ is a virtual numeraire for X. Then a continuous preference $\succsim$ on X satisfies (N.1) - (N.3) if and only if it admits a representation by a continuous, additive-equivariant utility.*

Additive-equivariance is a cardinal property, and does not hold for arbitrary representations of a preference. For example, Cobb-Douglas utility functions on $\mathbb{R}^L_{++}$, where the numeraire is defined by $\phi(\alpha, x) = e^{\alpha} x$ as in Example 4, are not additive-equivariant. Nevertheless, the underlying preferences admit an additive-equivariant representation for this numeraire (e.g., by taking the natural logarithm of the usual Cobb-Douglas utility).[21] Theorem 1 provides necessary and sufficient conditions, in terms of the ordinal structure of the underlying preference, for the existence of an additive-equivariant representation.

Remark 1. Continuity of the virtual numeraire and preference plays no role in the proof of Theorem 1 other than in verifying the continuity of the representation. A non-topological variant of the result remains true, where X is an arbitrary set equipped with an action ϕ. However, nothing can be said about the continuity of U. This is notable as topological assumptions are often crucial in ensuring the existence of a utility representation. Here, (N.1) - (N.3) alone suffice without any further stipulations on X or $\succsim$.[22]

Remark 2. For a fixed action ϕ, the representation in Theorem 1 is unique only up to an additive constant. However, note that for a given additive-equivariant U, $\beta > 0$ and $\gamma \in \mathbb{R}$, $\beta U + \gamma$ is an additive-equivariant representation of the same preferences, under the modified action defined

[21]See also Example 12 for an explicit computation of the additive-equivariant representation of von-Neumann Morgenstern preferences.

[22]Unlike continuity, the (linear) order structure of $\mathbb{R}_+$ plays an essential role in the proof. Many economically interesting classes of preferences are able to be expressed as invariant under either a partially or unordered monoid, however. For example, Cobb-Douglas preferences on $\mathbb{R}^L_+$ are characterized by invariance under the action of the partially ordered 'budget space,' see Trockel (1989). When $X = Z^{\mathbb{N}}$ denotes the space of discrete-time infinite-horizon consumption streams, the stationarity axiom of Koopmans (1960) amounts to invariance under an action on X of the free monoid generated by Z. It would be of interest to obtain a representation theorem for general invariant preferences that includes such examples.

by $\phi_\beta(\alpha, x) = \phi(\beta\alpha, x)$.[23] The action ϕ_β is equivalent to ϕ up to change of units; for example, if $\beta = 4$ and the numeraire is measured in dollars, then ϕ_β represents the same numeraire, but measured in quarters. Moreover, a preference is invariant under ϕ if and only if it is invariant under ϕ_β. Thus the failure of additive equivariance to be invariant up to positive affine transformations is a consequence of treating ϕ as a model primitive; up to choice of numeraire units, additive equivariance is preserved under positive affine transformations, as is standard. One implication of this, for example, is the failure of ρ to be identified in Example 3, see Fishburn and Rubinstein (1982).

3.4 Regularity Assumptions

We will henceforth impose the following three technical regularity conditions on model primitives, which, while not required for Theorem 1, are required in the sequel.

(A.1) **Injectivity**: For all $\alpha \in \mathbb{R}_+$, the map $\phi(\alpha, \cdot)$ is injective.[24]

We say that an alternative x is **reachable** from y, denoted $y \trianglelefteq x$ if there exists $\alpha \geq 0$ such that $x = \phi(\alpha, y)$. That is, if x is equal to y plus some additional numeraire. Let $\sim_\trianglelefteq$ denote the symmetric closure of this relation.[25] If ϕ satisfies (A.1), then $\sim_\trianglelefteq$ is an equivalence relation (see Lemma 1).

(A.2) **Cross Section**: There exists a continuous map $s : X/{\sim_\trianglelefteq} \to X$, such that, for all $y \in X/{\sim_\trianglelefteq}$,

$$(q \circ s)(y) = y,$$

where q is the quotient map taking X to $X/{\sim_\trianglelefteq}$, which carries its quotient topology.

(A.3) **No Loitering**: For all $x \in X$, there exists $\varepsilon > 0$ and $T > 0$ such that, for all $x' \in B_\varepsilon(x)$ and all $\alpha > T$:

$$\phi(\alpha, x') \notin B_\varepsilon(x),$$

where $B_\varepsilon(x)$ denotes the ε-ball about x.

[23] In particular, the group of order-preserving monoid isomorphisms of $\mathbb{R}_+$ to itself under composition is isomorphic to the multiplicative group of positive reals. See, for example Fuchs (2011).

[24] If the set of preferences that satisfy (N.1) - (N.3) is non-empty, then for all $x \in X$, $\phi(\cdot, x)$ is necessarily injective: if for some $x \in X$ and $\alpha < \beta$

$$\phi(\alpha, x) = \phi(\beta, x),$$

then $\phi\big(\beta - \alpha, \phi(\alpha, x)\big) = \phi(\alpha, x)$, but as $\beta - \alpha > 0$, every reflexive relation on X violates (N.2).

[25] Recall the symmetric closure of a relation R is the smallest symmetric relation containing R.

Injectivity simply requires that there not be any pair of distinct alternatives x and y that become *equivalent* after being combined with a sufficient, common quantity of numeraire. Thus the process of adding numeraire is, in principle, reversible. Cross section is a weak technical assumption that ensures that at least some of the indifference curves of any preference satisfying (N.1) - (N.3) are sufficiently connected.[26] Absent it, it could be the case that every indifference curve misses some equivalence classes of $\sim_{\trianglelefteq}$. Finally, no loitering ensures that no alternative can be regarded as the result of adding infinite numeraire to any other.

4 Data & Elicitiation

An **experiment** is a finite collection $\mathcal{E}$ of pairs of alternatives such that no two alternatives (belonging even to differing pairs) are related under $\trianglelefteq$.[27] For a given agent and pair $\{x, y\} \in \mathcal{E}$, we will assume an observation of both (i) which alternative in $\{x, y\}$ is (weakly) more preferable than the other, and (ii) how much virtual numeraire is needed, in addition to receiving the less preferable alternative, to make the agent indifferent with receiving the more preferable alternative.[28] That is, we assume we observe the compensation difference between the less and more favorable alternatives.

We suppose a data set consisting of $N \geq 1$ repetitions of such an experiment. Formally, let $\vec{\mathcal{E}}$ denote the collection of all ordered pairs (x, y) such that $\{x, y\} \in \mathcal{E}$. A **data set** is a collection $\{Y^n\}_{n=1}^N$ of vectors in $\mathbb{R}^{\vec{\mathcal{E}}}$, where, for each $(x, y) \in \vec{\mathcal{E}}$:

$$Y_{xy}^n = \begin{cases} \alpha & \text{if } \phi(\alpha, x) \sim_n y \\ -\alpha & \text{if } \phi(\alpha, y) \sim_n x, \end{cases}$$

where $\phi(\alpha, x) \sim_n y$ denotes that α is the compensation difference between x and y in the n-th repetition. Since $Y_{xy}^n = -Y_{yx}^n$, we may identify the space of all possible data sets with $\mathbb{R}^{\mathcal{E}}$ by fixing a choice of ordering for each pair. When $N = 1$, a data set corresponds simply to observing a finite

[26]For an example of an X and ϕ which do not satisfy this condition, see Appendix F.

[27]The assumption that no pairs of alternatives $\{x, y\} \in \mathcal{E}$ are $\trianglelefteq$-related amounts to not inquiring how much numeraire would make a subject indifferent between receiving x versus x plus α units of numeraire. The stronger requirement that no two of alternatives belonging even to different pairs are $\trianglelefteq$-related is purely for convenience. It may be dropped, at the cost of requiring a slight modification to our our rationalizability condition. See Section 5.1.

[28]This is the numerical quantity $\alpha \geq 0$ such that

$$\phi(\alpha, x) \sim y,$$

when $y \succsim x$.

set of compensation differences for a single agent. Unless otherwise specified, we interpret the case of $N > 1$ as corresponding to observations of a sample of N agents from some fixed population.[29] In Section 6 we will consider how to test hypotheses about the expected behavior of such a population.

4.1 Elicitation

In this section we will present a dominant-strategy incentive-compatible mechanism to truthfully elicit compensation differences data. Our approach may be seen as a generalization of Becker et al. (1964). We will consider the elicitation problem for a given observation, and extend to a full experiment via lottery.[30] Let $\{x, y\} \in \mathcal{E}$ be an arbitrary pair of alternatives. We first define two intermediate mechanisms: in the x-mechanism, the agent is offered the opportunity to submit a non-negative 'sell price' in numeraire units for x, denoted s, to a computerized buyer. The buyer simultaneously and blindly selects a non-negative 'buy' price b. If $s > b$, no trade occurs and the agent is awarded x. If $b \geq s$, then a trade occurs, and instead of x, the agent receives $\phi(b, y)$. We analogously define the y-mechanism. Compensation differences may then be elicited by presenting the subject with a choice: they are invited to submit a sell price in either the x- or y-mechanism, but not both. However, in whichever mechanism they do not choose, a sell price of 0 will be submitted on their behalf. After the bids have been submitted, a coin is flipped to select either x or y, and the associated mechanism's reward is allocated to the agent, regardless of which intermediate mechanism they chose to manually submit a sell price for.

We model the agent's decision problem using the states of the world formalism. We do so to highlight that the incentive-compatibility of our mechanism does not depend on the manner in which the subject handles probabilities. Suppose that $\Omega = \mathbb{R}^2_+ \times \{x, y\}$ denotes the payoff-relevant states of the world; the tuple (b_x, b_y, z) denotes the state in which the computer selects bids b_x in the x-mechanism, b_y in the y-mechanism, and the payoff-determining mechanism is $z \in \{x, y\}$. A choice of action for the agent consists of a tuple in $\{x, y\} \times \mathbb{R}_+$, corresponding a choice of which intermediate mechanism to participate in, and what sell price to submit there. Let X^* denote the set maps from $\Omega \to X$ that are awarded by this mechanism. We assume the agent has preferences

[29] We note such data may alternatively be interpreted as repeated observations of the noisy preference(s) of a single agent.

[30] Stemming back to Holt (1986) there has been concern that, in theory, random-lottery incentive systems rely on implicit assumptions about choice under uncertainty that may be problematic. However, there is a wide range of empirical evidence suggesting that these concerns do not bear out in practice, e.g. Starmer and Sugden (1991), Hey and Lee (2005), and Lee (2008).

$\succsim^*$ over X^* and say these are **consistent** with their preference $\succsim$ over X if, for all $f, g \in X^*$, $f(\omega) \succsim g(\omega)$ for all $\omega \in \Omega$ implies $f \succsim^* g$.

Theorem 2. *Suppose an agent has preferences $\succsim$ on X that satisfy (N.1) - (N.3), and preferences $\succsim^*$ over X^* that are consistent with $\succsim$. Then choosing to submit a bid equal to their true compensation difference, in the mechanism corresponding to the more-preferred alternative, is $\succsim^*$-optimal.*

Remark 3. The assumption that the agent had a well-defined preference relation $\succsim^*$ over X^* is not required for the result. Even if $\succsim^*$ is a highly incomplete and non-transitive relation, Theorem 2 remains valid so long as $\succsim^*$ remains consistent in the above sense with $\succsim$.

5 Goodness of Fit

An experiment $\mathcal{E}$ may be associated with an undirected graph $(\mathcal{V}, \mathcal{E})$ whose vertices are those alternatives featuring in the experiment, and whose edges are the pairs defining the experiment:

$$\mathcal{V} = \bigcup_{\{x,y\} \in \mathcal{E}} \{x, y\}.$$

We will assume henceforth that $(\mathcal{V}, \mathcal{E})$ is always a connected graph. A **flow** on $(\mathcal{V}, \mathcal{E})$ is a skew-symmetric, real-valued function on $\vec{\mathcal{E}}$.[31] Given a data set $\{Y^n\}_{n=1}^N$, let $\bar{Y} = \frac{1}{N} \sum Y^n$. For a given agent, Y^n_{xy} is the compensation difference between x and y for agent n, hence $\bar{Y}_{xy}$ is precisely the sample average, or (empirical) expected compensation difference between x and y. Thus, for any experiment, $\bar{Y}$ defines a flow on $(\mathcal{V}, \mathcal{E})$. Conversely, every flow may be regarded as arising from some data set for $\mathcal{E}$.

5.1 Rationalizability

We say that a data set $\{Y^n\}_{n=1}^N$ is **cardinally consistent** if, for every finite sequence (x_0, x_1), (x_1, x_2), $\ldots$, $(x_L, x_0) \in \vec{\mathcal{E}}$,

$$\sum_{l=0}^{L} \bar{Y}_{x_l x_{l+1}} = 0, \tag{1}$$

where subscripts are understood mod-L. By minor abuse of notation, we will also refer to a flow such as $\bar{Y}$ as being cardinally consistent if (1) holds. Cardinal consistency amounts to a flow belonging to the kernels of a finite collection of linear functionals, thus the sub-collection of cardinally consistent flows forms a linear subspace of the space of all flows.

[31] That is, $F : \vec{\mathcal{E}} \to \mathbb{R}$ is a flow if and only if, for all $(x, y) \in \vec{\mathcal{E}}$, $F_{xy} = -F_{yx}$.

A data set is **rationalized** by a continuous, additive-equivariant preference if there exists a continuous preference relation $\succsim$ on X that satisfies (N.1) - (N.3), such that for all $(x, y) \in \vec{\mathcal{E}}$ with $\bar{Y}_{xy} \geq 0$:

$$\bar{Y}_{xy} = \alpha \iff \phi(\alpha, x) \sim y.$$

Similarly, we say $\{Y^n\}_{n=1}^N$ is rationalized by an additive-equivariant utility U if, for all $(x, y) \in \vec{\mathcal{E}}$:

$$\bar{Y}_{xy} = U(y) - U(x).$$

By Theorem 1 these properties coincide, and we are justified in speaking of the additive-equivariant rationalizability of the data. When the data contain observations of a sample population of agents, additive-equivariant rationalizability refers to whether the sample population is rationalizable in expectation. It is clear that if $\bar{Y}$ is rationalizable by an additive-equivariant utility, it will necessarily be cardinally consistent. Our next result establishes that cardinal consistency is also sufficient.

Theorem 3. *Let (X, ϕ) satisfy (A.1) - (A.3), and suppose the set of continuous preferences satisfying (N.1) - (N.3) is non-empty. Then for every experiment $\mathcal{E}$, for any dataset, the following are equivalent:*

(i) The data are cardinally consistent.

(ii) The data are rationalizable by a continuous preference satisfying (N.1) - (N.3).

(iii) The data are rationalized by a continuous, additive-equivariant utility.

Theorem 3 characterizes the testable implications of additive-equivariance for any experiment. It also highlights a benefit of additive-equivariant preferences: testing rationalizability amounts to investigating whether or not the data $\bar{Y}$ lies in a fixed, linear subspace that is explicitly determined by the structure of the experiment $\mathcal{E}$.

5.2 'Best Fit' Approximations

In the preceding section we noted that, for any experiment, the subset of cardinally consistent data form a linear subspace of the vector space of flows on $(\mathcal{V}, \mathcal{E})$. We first provide an alternative characterization of this subspace. For a given experiment, let $\mathcal{F} \subsetneq \mathbb{R}^{\vec{\mathcal{E}}}$ denote the vector space of all flows on $(\mathcal{V}, \mathcal{E})$, and let $\mathcal{U}$ denote the set of all utility functions over the vertices $\mathcal{V}$:

$$\mathcal{U} = \{u : \mathcal{V} \to \mathbb{R}\}.$$

To any utility function $u \in \mathcal{U}$ one may associate its **gradient**, a flow whose value on a given oriented edge is given by the signed difference of the utility values at its endpoints:

$$(\operatorname{grad} u)_{xy} = u_y - u_x$$

This defines a linear map $\operatorname{grad} : \mathcal{U} \to \mathcal{F}$. The following two propositions are standard, but we provide proofs for completeness.

Proposition 1. *A flow $F \in \mathcal{F}$ is cardinally consistent if and only if it belongs to the image of the gradient.*

In light of Theorem 3, the data $\bar{Y}$ are rationalizable by an additive-equivariant utility if and only if $\bar{Y}$ is the discrete gradient of a utility function $u \in \mathcal{U}$.[32] Fix an ordering of $\mathcal{V} = \{v_1, \ldots, v_K\}$. By minor abuse of notation we will write i for v_i, F_{ij} for $F_{v_i v_j}$, and so forth when no confusion will result. Any flow is uniquely determined by its values on oriented edges $(i,j) \in \vec{\mathcal{E}}$ with $i < j$. Thus we identify $\mathcal{F}$ with $\mathbb{R}^{\mathcal{E}}$, with basis $\left\{\mathbb{1}_{(i,j)}\right\}_{\{(i,j)\in\vec{\mathcal{E}}: i<j\}}$.[33] Using this basis, we endow $\mathcal{F}$ with an inner product via:

$$\langle F, F' \rangle = \sum_{\{(i,j)\in\vec{\mathcal{E}}\,:\, i<j\}} F_{ij} F'_{ij}.$$

The **divergence** of a flow is the real valued function on vertices defined by:

$$(\operatorname{div} F)_i = \sum_{j \in N(i)} F_{ij},$$

where $N(i) \subseteq \mathcal{V}$ denotes the set of neighbors of v_i. In other words, the divergence computes the vector of net outflows minus inflows at each vertex. This defines a linear map $\operatorname{div} : \mathcal{F} \to \mathcal{U}$, and when $\mathcal{U}$ carries its standard inner product, $-\operatorname{div}$ is the adjoint of the gradient operator.

Proposition 2. *For any experiment, the space of flows on $(\mathcal{V}, \mathcal{E})$ admits an orthogonal direct-sum decomposition as:*[34]

$$\mathcal{F} = \operatorname{im}(\operatorname{grad}) \oplus \ker(\operatorname{div}).$$

[32]This highlights a recurring parallel between the discrete graph theoretic methods employed here and the differential calculus. In particular, cardinal consistency is a discrete analogue of the requirement that a gradient vector field integrate to zero around every closed curve in a domain.

[33]For example, let $(\mathcal{V}, \mathcal{E})$ denote the complete graph on three vertices. We identify a flow F with the vector (F_{01}, F_{02}, F_{12}). If F is a cyclic flow of α units from v_0 to v_1, v_1 to v_2, and v_2 to v_0, then we identify F with $(\alpha, -\alpha, \alpha)$.

[34]In differentiable terms, Proposition 2 is just a graph theoretic rephrasing of the Helmholtz decomposition of vector calculus.

By Proposition 1, the image of the gradient consists precisely of the cardinally consistent hence additive-equivariant rationalizable flows. Call a flow $C \in \mathcal{F}$ a **perfect cycle** if it has vanishing divergence, and is supported on a single loop in $(\mathcal{V}, \mathcal{E})$.[35] The kernel of the divergence is precisely the span of the perfect cycles in $\mathcal{F}$. Thus $\bar{Y}$ admits a unique decomposition into a cardinally consistent term, and a 'purely inconsistent' term, expressible as a sum of perfect cycles.[36] For a given $\bar{Y}$, we define $\hat{Y}$, the **best fit** cardinally consistent approximation to $\bar{Y}$ as the (orthogonal) projection of $\bar{Y}$ onto im(grad). By Proposition 2, this may be computed by solving the following least squares problem:

$$\min_{u \in \mathcal{U}} \left\| (\operatorname{grad} u) - \bar{Y} \right\|_2^2. \tag{2}$$

We motivate our choice of $\hat{Y}$ as best fit with the following informal axiomatization. Firstly, given our interest in the rationalizability of the *average* preference of a population, it is desirable that the best-fit for a sample average $\bar{Y}$ should correspond to the average of the best fits of each agent in the sample. In particular, the map taking $\bar{Y}$ to the best fitting cardinally consistent approximation should be linear. Secondly, if $\bar{Y}$ is cardinally consistent, then $\bar{Y}$ should always be its own best fit, hence the map should additionally be idempotent. Finally, by Proposition 1, every cardinally consistent flow corresponds to a unique (up to a constant of integration) vector of utilities $u \in \mathcal{U}$ and hence to an implicit ranking of the alternatives $\mathcal{V}$. It is desirable that this ranking be invariant under permuting the labels of the vertices; in particular every perfect cycle should map to indifference. Then by linearity, the best fit for any flow in ker(div) should be 0, and this uniquely identifies our selection of $\hat{Y}$.

5.2.1 Relation to Social Choice

Any (cardinal) social welfare function that takes in a data set $\{Y^n\}_{n=1}^N$ and returns a utility function over $\mathcal{V}$ defines a notion of selecting for a best-fit approximation. It is known that solutions to (2) correspond to a natural cardinal generalization of the Borda score of social choice theory.[37] Suppose momentarily that $(\mathcal{V}, \mathcal{E})$ is a complete graph. We define the *cardinal Borda score* for such a data

[35] A loop in $(\mathcal{V}, \mathcal{E})$ is a connected subgraph such that every vertex is contained in precisely two edges. A flow is supported on a loop if takes the value zero on every edge that does not belong to the loop.

[36] Though the decomposition of the inconsistent term into a sum of perfect cycles will not be unique, see, e.g., Figure 2.

[37] The observations of this section are implicit in Young (1974), and noted explicitly in, e.g., Jiang et al. (2011) in the context of aggregating online rankings.

set via:

$$s_{\mathrm{CB}}(v_i) = \frac{1}{K}\sum_{j\neq i}\bar{Y}_{ji},$$

where the $1/K$ is a positive coefficient that does not affect the ranking.[38] Writing A for the adjacency matrix of $(\mathcal{V},\mathcal{E})$ and D for the diagonal matrix with $D_{ii} = \deg(v_i)$, the Laplacian matrix of $(\mathcal{V},\mathcal{E})$ is defined as $L = D - A$. Then the normal equation for (2) may be written as:

$$L\,u = -\operatorname{div}\bar{Y}. \tag{3}$$

Since we have assumed that $(\mathcal{V},\mathcal{E})$ is connected, the kernel of the Laplacian is spanned by the vector $(1,\ldots,1)$, and thus every solution to (3) will be unique only up to addition of a constant function. In particular, we may for determinacy's sake select the minimum norm solution for a given $\bar{Y}$; this is equivalent to assuming that $\sum_{i=1}^{K} u_i = 0$. Equation (3) says that any solution u is determined by a strong 'averaging' property. In particular, for any v_i:

$$u_i = \frac{1}{\deg(i)}\left[\sum_{j\in N(i)} u_j + \bar{Y}_{ji}\right] \tag{4}$$

so u_i is an unweighted average of the utility values of each neighboring v_j, plus the observed flow from each neighboring v_j to v_i. If $(\mathcal{V},\mathcal{E})$ is complete, then the unique (zero-sum) utility vector satisfying (4) for each v_i is precisely the cardinal Borda score. In this instance, (4) becomes:

$$u_i = \frac{1}{K-1}\left[\sum_{j\neq i} u_j + \bar{Y}_{ji}\right].$$

But, as $\sum_{j\neq i} u_j = -u_i$, this simplifies to:

$$u_i = \frac{1}{K}\sum_{j\neq i}\bar{Y}_{ji},$$

or the cardinal Borda score. More generally, the solution to (2) is wholly determined by this averaging property, which is the extension of the characteristic feature of the cardinal Borda score from complete experiments to incomplete ones.

[38] This notation is justified as, when $(\mathcal{V},\mathcal{E})$ is complete, the Borda count would simply be:

$$s_{\mathrm{B}}(v_i) = \sum_{j\neq i}\sum_{n=1}^{N}\operatorname{sign}(Y_{ji}^{n}),$$

as Y_{ji}^{n} is positive if $v_i \succ v_j$, and negative if $v_j \succ v_i$. Thus s_{B} precisely totals 'net votes for v_i over v_j' across all $j \neq i$. The cardinal Borda count allows the strength of preference, reflected in the magnitudes of Y_{ji}^{n}, to factor in.

5.2.2 Quantifying Inconsistency in the Residual

By Proposition 2, the residual $R \equiv \bar{Y} - \hat{Y}$ will belong to the kernel of the divergence operator, and thus consists of a sum of perfect cycles. In this section, we will show there is a natural economic measure of the 'irrationality' of any such residual flow: its L^1-norm. Suppose first, that R is a perfect cycle. Then for some finite sequence $(v^0, v^1), (v^1, v^2), \ldots, (v^L, v^0) \in \vec{\mathcal{E}}$, we have $R_{v^l v^{l+1}} = \bar{c} \geq 0$. Here, we have used superscript indices to distinguish them from the enumeration used to define the basis for $\mathcal{F}$, which we will always denote with a subscript.[39] If we interpret this residual as a data set arising from a single (representative) agent, then for this agent, for all l:

$$\phi(\bar{c}, v^l) \sim v^{l+1},$$

where the superscripts are understood mod-L. For every l, this agent would then be willing to trade v^l, plus up to $\bar{c}$ units of numeraire, for v^{l+1}. A savvy arbitrageur could exploit such an agent as a 'numeraire pump,' and extract $\|R\|_1 = \bar{c}\,(L+1)$ units of numeraire from the agent via a cyclic sequence of trades.[40] This suggests a numeraire-valued analogue of the classic money pump index as a natural measure of the degree of inconsistency for such residuals.[41] When R is a pure cycle, we define the **money pump** value of R as:

$$MP(R) = \big\|R\big\|_1 = \sum_{\{(i,j)\in\vec{\mathcal{E}}\,:\, i<j\}} \big|R_{ij}\big| = \bar{c}\,(L+1).$$

To extend from residuals consisting of a single pure cycle to all of ker(div), it is natural to decompose a general $R \in \ker(\mathrm{div})$ into a sum of pure cycles, and compute the sum of the money pumps of each of these component cycles. However, such a decomposition into pure cycles will be non-unique, and the sum of the money pumps of two different decompositions of the same residual will in general differ, see Figure 2. Thus we consider the most conservative extension. Let $\mathfrak{C} \subsetneq \mathcal{F}$ denote the set of pure cycles. For any $R \in \ker(\mathrm{div})$ let $\mathfrak{D}(R)$ denote the collection of all finite decompositions of R into pure cycles, i.e. those collections $\{C_1, \ldots, C_M\} \subseteq \mathfrak{C}$ such that $\sum_m C_m = R$. We extend $MP : \mathfrak{C} \to \mathbb{R}$ to a function $MP^* : \ker(\mathrm{div}) \to \mathbb{R}$ via:

$$MP^*(R) = \inf_{\{C_1,\ldots,C_M\}\in\mathfrak{D}(R)} \sum_{m=1}^{M} MP(C_m).$$

[39] Thus, in particular, $R_{v^l v^{l+1}}$ here corresponds to the flow along a particular edge in this cycle.

[40] Implicit in such a narrative is the assumption that after one sequence of 'trades around the cycle' the agent wises up and trades no more.

[41] See, e.g., Echenique et al. (2011) and the references therein.

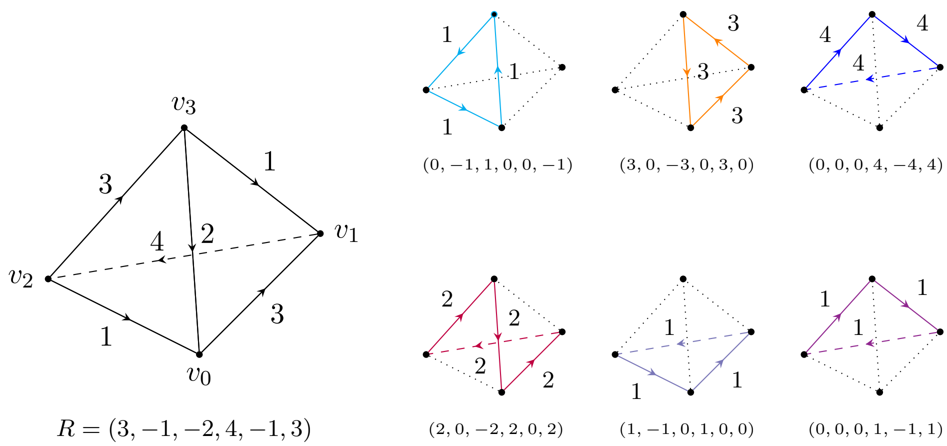


Figure 2: A residual flow $R = (R_{01}, R_{02}, R_{03}, R_{12}, R_{13}, R_{23})$, belonging to the kernel of the divergence, along with two decompositions into sums of perfect cycles. The lower bound of $\lVert R \rVert_1 = 14$ is attained by sum of the money pump values of the bottom (though not the top) decomposition.

Our next result shows that, in spite of its definition as a value function, MP^* is real valued, has a simple, closed-form expression, and is indeed an extension of MP.

Proposition 3. *For all $R \in \ker(\mathrm{div})$, the money pump value of R is given by:*

$$MP^*(R) = \sum_{\{(i,j) \in \vec{\mathcal{E}} : i<j\}} \lvert R_{ij} \rvert = \lVert R \rVert_1.$$

Moreover, the infimum in the definition of MP^* is always attained by some decomposition in $\mathfrak{D}(R)$.

5.3 Shape Constraints

One is often times concerned not with simply the additive-equivariant rationalization of some $\bar{Y}$ but rather when it is rationalizable by an additive-equivariant utility possessing additional properties such as (quasi-)concavity, monotonicity, homogeneity etc. These constraints are able to be encoded into the problem via the consideration of a constraint set:

$$\mathcal{K} = \{u \in \mathcal{U} : \exists \psi \text{ possessing the desired structure s.t. } \psi(v_i) = u_{v_i}\}.$$

Formally, we say a set $\mathcal{K} \subseteq \mathcal{U}$ defines a set of **shape constraints** if (i) $\mathcal{K}$ is convex, and (ii) $\mathcal{K} + \mathrm{span}\{(1, \ldots, 1)\}$ is closed.[42] Rather than solve an unconstrained least squares problem as in

[42]This closure condition is innocuous and satisfied under all economically interesting cases we are aware of. It serves only to guarantee that the image $\mathrm{grad}(\mathcal{K})$ is a closed convex subset of $\mathcal{F}$ for any (connected) experiment. For

(2), one projects instead onto grad($\mathcal{K}$):

$$\min_{u \in \mathcal{K}} \left\| (\operatorname{grad} u) - \bar{Y} \right\|_2^2. \tag{5}$$

By Proposition 2, the residual from (5) decomposes into two components: one with vanishing divergence and one that is cardinally consistent. The magnitude of the first component still reflects how well additive-equivariance is borne out in the data. The magnitude of the latter captures the shadow price, in model fit terms, of including those axioms which correspond to the binding constraints at the solution to (5).

When a family of models are defined on a common domain X and are additively-equivariant relative to the same virtual numeraire ϕ, equation (5) allows for a simple, tractable approach to model selection. Given a family of models, and hence shape constraints $\{\mathcal{K}_m\}_{m=1}^M$, computing (5) for each K_m yields an model-specific measure of how consistent the data is with model m. By comparing these values, one obtains a ranking of these models, given $\{Y^n\}$.

One often encounters models that are strict generalizations of each other. For example, the variational preferences of Maccheroni et al. (2006) generalize the multiple priors model of Gilboa and Schmeidler (1989) by relaxing the homogeneity of the utility functional. In this case, $\mathcal{K}_{MEU} \subseteq \mathcal{K}_{VAR}$ and hence the value of (5) will be at least as high for the multiple priors model as it is for variational preferences. However, the difference in values nonetheless allows one to surmise how well supported the marginal axioms (here homogeneity of the utility functional) are in the data. If this difference is large, it suggests that the shadow price, in terms of mean squared error, of including those particular axioms is large.

5.3.1 Examples

In many economic applications, evaluating (5) amounts to solving a quadratic program with linear constraints; to both illustrate this, as well as to provide a catalogue of ready-to-use examples, in Appendix E we provide a shape constraint 'cookbook,' consisting of explicit characterizations (including proofs) of the shape constraints for a number of different properties and models, including those appearing in this section.

such an experiment, ker(grad) is just the diagonal of $\mathcal{U}$ hence this condition is just a requirement $\mathcal{K} + \ker(\text{grad})$ be closed, which is holds if and only if grad($\mathcal{K}$) is closed; see, e.g., Holmes (2012), Lemma 17.H.

Example 7 (Quasilinearity Revisted)**.** Consider the setting of Example 1, where $X = \mathbb{R}^2_+$, and $\phi\big(\alpha, (x, y)\big) = (x + \alpha, y)$. Suppose now we wish to test whether or not the data is consistent with preferences representable by a quasilinear utility:

$$U(x, y) = v(y) + x,$$

where additionally v is increasing and concave. Let $\mathcal{K}_{QIC}$ denote the set of vectors in $\mathcal{U}$ that are restrictions of quaslinear (in the first variable), increasing, and concave functions. Then solving (5) with $\mathcal{K} = \mathcal{K}_{QIC}$ is equivalent to solving:

$$\begin{aligned}
\min_{u \in \mathcal{U}} \quad & \big\|(\operatorname{grad} u) - \bar{Y}\big\|_2^2 & \\
\text{subject to} \quad & u_i = \langle \pi_i, v_i \rangle + \gamma_i & \forall i = 1, \dots, K \\
& \langle \pi_i, v_i \rangle + \gamma_i \leq \langle \pi_j, v_i \rangle + \gamma_j & \forall i, j = 1, \dots, K \\
& \pi_i^1 = 1 & \forall i = 1, \dots, K \\
& \pi_i \geq 0 & \forall i = 1, \dots, K
\end{aligned} \tag{6}$$

for $u \in \mathbb{R}^K$ and, for all $i = 1, \dots, K$, $\pi_i \in \mathbb{R}^2$, $\gamma_i \in \mathbb{R}$ (where π_i^1 denotes the first component of π_i). This equivalence rests on the observation that, for any quasilinear, increasing and concave U, forming a polyhedral concave function by taking the pointwise minimum of the collection of supporting hyperplanes of U at each v_i is indistinguishable from the true U. The conditions then that the supergradients π_i have first component 1 and belong to the non-negative orthant encode, respectively, the quasilinearity and monotonicity constraints.

Sometimes, it is necessary to first perform a change of coordinates to bring the class of utilities of interest into additive-equivariant form. Such transformations must be accounted in how they apply to the values v_i.

Example 8. (Cobb-Douglas Preferences) Let $X = \mathbb{R}^L_{++}$ and $\phi(\alpha, x) = e^\alpha x$. Though Cobb Douglas preferences satisfy (N.1) - (N.3), Cobb Douglas utility functions of the form:

$$U(x) = \prod_{i=1}^{L} x_i^{\beta_i} \tag{7}$$

where $\langle \beta, \mathbb{1}_L \rangle = 1$, are not not additive-equivariant. Consider the following change of coordinates:

$$H(x_1, \dots, x_L) = \big(\ln x_1, \dots, \ln x_L\big),$$

where $H : X \to \mathbb{R}^L$. Then, taking natural logs of (7),

$$\begin{aligned}\ln U\big(\phi(\alpha, x)\big) &= \langle \beta, \alpha + H(x) \rangle \\ &= \langle \beta, H(x) \rangle + \alpha\end{aligned}$$

as desired. Thus for purposes of regression, we treat $H(X)$ as our consumption space, rather than X.[43] Then computing (5) for Cobb-Douglas preferences, in these coordinates, amounts to solving the following quadratic program with polyhedral constraint:

$$\begin{aligned} \min_{u \in \mathcal{U}} \quad & \big\|(\operatorname{grad} u) - \bar{Y}\big\|_2^2 \\ \text{subject to} \quad & u_i = \big\langle \beta, H(v_i) \big\rangle \quad \forall i = 1, \ldots, K \\ & \langle \beta, \mathbb{1}_L \rangle = 1 \\ & \beta \geq 0 \end{aligned} \tag{8}$$

for $\beta \in \mathbb{R}^L$. The second constraint encodes the standard normalization that the Cobb-Douglas exponents sum to one, and implies the full utility will be identified from any solution (u^*, β^*).

Example 9 (Risk-neutral Variational Preferences)**.** Let S be a finite set of states of the world, and $X = \mathbb{R}^S$. Let $\phi(\alpha, x) = x + \alpha \mathbb{1}_S$, and suppose $\mathcal{V} = \{v_1, \ldots, v_K\}$, where $v_K = 0$. We say that a utility function $U : X \to \mathbb{R}$ is a variational utility functional if it is of the form:

$$U(x) = \min_{\pi \in \Delta(S)} \langle \pi, x \rangle + c(\pi),$$

where $c : \Delta(S) \to [0, \infty]$ is convex, lower-semicontinuous, and attains its minimal value 0. Let $\mathcal{K}_{\text{VAR}}$ denote the subset of $\mathcal{U}$ consisting of restrictions of variational utility functionals to $\mathcal{V}$. Then equation (5), for $\mathcal{K} = \mathcal{K}_{\text{VAR}}$, is equivalent to:

$$\begin{aligned} \min_{u \in \mathcal{U}} \quad & \big\|(\operatorname{grad} u) - \bar{Y}\big\|_2^2 \\ \text{subject to} \quad & u_i = \langle \pi_i, v_i \rangle + \gamma_i & \forall i = 1, \ldots, K \\ & \langle \pi_i, v_i \rangle + \gamma_i \leq \langle \pi_j, v_i \rangle + \gamma_j & \forall i, j = 1, \ldots, K \\ & \langle \pi_i, \mathbb{1}_S \rangle = 1 & \forall i = 1, \ldots, K \\ & \pi_i \geq 0 & \forall i = 1, \ldots, K \\ & \gamma_K = 0, \end{aligned} \tag{9}$$

[43] Formally, $\mathbb{R}_L$ carries the 'induced action' $\tilde{\phi} : \mathbb{R}_+ \times \mathbb{R}_L \to \mathbb{R}_L$ given by:

$$\phi(\alpha, \tilde{x}) = x + \alpha \mathbb{1}_L,$$

for all $\tilde{x} \in \mathbb{R}_L$.

for $\pi_1 \dots, \pi_K \in \mathbb{R}^S$ and $\gamma_1, \dots, \gamma_K \in \mathbb{R}$. The first two sets of constraints are simply a system of supergradient inequalities. If one additionally imposes $\gamma_i = 0$ for all $i = 1, \dots, K$, the resulting constraint set characterizes regression for the risk-neutral analogue of the maxmin preferences of (Gilboa and Schmeidler, 1989). Additionally, by further restricting the π_i one obtains constraint sets for risk-neutral variants of the convex Choquet expected utility theory of Schmeidler (1989) and subjective expected utility Anscombe and Aumann (1963). See Appendix E for these and other examples.

Remark 4. It is possible to extend Example 9 beyond risk neutrality. Suppose instead the experimenter wishes to test whether or not preferences over monetary acts are representable by a utility of the form:

$$U(x) = \min_{\pi \in \Delta(S)} \mathbb{E}_\pi \big[\tilde{u}(x)\big] + c(\pi), \tag{10}$$

where $\tilde{u}$ is a fixed, increasing, and unbounded above Bernoulli utility $\tilde{u} : \mathbb{R} \to \mathbb{R}$.[44] Such a $\tilde{u}$ could be chosen, for example, on the basis of a first-stage estimation procedure, or based on theoretic considerations.[45] Let $\phi_{\tilde{u}}$ denote the action on $\mathbb{R}^S$ defined component-wise via $x_s \mapsto \tilde{u}^{-1}\big(\tilde{u}(x_s) + \alpha\big)$. For any x, $\phi_{\tilde{u}}(\alpha, x)$ is the monetary act which yields, under $\tilde{u}$, precisely α additional utility in each state. For compensation differences data measured using $\phi_{\tilde{u}}$, one can test the shape constraints corresponding to (10) exactly as in (9), but replacing each v_i instead with the corresponding vector of utilities under $\tilde{u}$. For ease of exposition, in Appendix E we present shape constraint characterizations of various utility functionals on $\mathbb{R}^S$ for the risk-neutral case (i.e. where $\tilde{u}$ is identity). However, all characterizations may be adapted in this manner to allow for more general choices of $\tilde{u}$.

Finally, many other economically interesting properties constitute valid sets of shape constraints. For example, additive separability, given some numeraire with respect to which the preference is invariant, forms a shape constraint.

Example 10. (Additive Separability as Shape Constraint): Suppose $X = X_1 \times X_2$ where X_1 and X_2 are metric spaces, and ϕ is a virtual numeraire. For ease of exposition, suppose that ϕ only acts on X_1 (i.e. the second coordinate of $\phi\big(\alpha, (x_1, x_2)\big)$ is x_2). We say an additive-equivariant utility

[44] It is straightforward to adapt to the case where $\tilde{u} : \mathbb{R}_+ \to \mathbb{R}$.

[45] Intuitively, the experiment outlined in Example 9 concerns only 'horse race' lotteries and hence is unable to simultaneously estimate attitudes towards 'roulette' lotteries, necessitating $\tilde{u}$ being supplied exogenously.

representation $U : X \to \mathbb{R}$ is additively separable if:

$$U(x_1, x_2) = W^1(x_1) + W^2(x_2).$$

Let Π_1, Π_2 denote the projections onto X_1 and X_2 respectively. Suppose that $\left|\Pi_1(\mathcal{V})\right| = V_1$ and $\left|\Pi_2(\mathcal{V})\right| = V_2$. Any function $w^1 : V_1 \to \mathbb{R}$ may, without loss of generality be regarded as a K-vector, where $w^1_i = (w \circ \Pi_1)(v_i)$, and likewise V_2.[46] Suppose, finally, that every pair of elements in $\Pi_1(\mathcal{V})$ are $\sim_{\trianglelefteq}$-unrelated. Then evaluating (5) where $\mathcal{K}$ is the set of restrictions of additively separable functions, is equivalent to solving:

$$\begin{aligned} \min_{u \in \mathcal{U}} \quad & \left\|(\operatorname{grad} u) - \bar{Y}\right\|_2^2 \\ \text{subject to} \quad & u_i = w^1_i + w^2_i && \forall i = 1, \ldots, K \\ & w^1_i = w^1_j && \forall i, j \in 1, \ldots, K \text{ s.t. } \Pi_1(v_i) = \Pi_1(v_j) \\ & w^2_i = w^2_j && \forall i, j \in 1, \ldots, K \text{ s.t. } \Pi_2(v_i) = \Pi_2(v_j) \end{aligned} \tag{11}$$

for $u, w^1, w^2 \in \mathbb{R}^K$. Moreover, it is straightforward to generalize this example by relaxing the assumptions on the form of the action, as well as to include more than two factors, which is another advantage of the cardinal nature of compensation differences data.

6 Testing

In this section, we consider a stochastic analogue of our regression framework. We are interested in testing the hypothesis that a population of agents has preferences that are, in expectation, representable by some additive-equivariant utility U (possibly satisfying some collection of shape constraints). Formally, we assume an underlying linear model where for each $(i, j) \in \vec{\mathcal{E}}$, we observe the underlying 'true' population compensation difference Y^0_{ij}, polluted by a mean-zero, individual-specific shock.

> **Data Generating Process**: For all $\{x, y\} \in \mathcal{E}$ there is fixed, non-stochastic compensation difference $Y^0_{xy} = -Y^0_{yx}$. The data $\{Y^n\}_{n=1}^N$ is a random sample of N independent draws of the random flow Y, where for each $(i, j) \in \vec{\mathcal{E}}$ with $i < j$:
>
> $$Y_{ij} = Y^0_{ij} + \epsilon_{ij},$$
>
> where (i) $\mathbb{E}(\epsilon_{ij}) = 0$, and (ii) $\operatorname{Var}(\epsilon_{ij}) < +\infty$.[47]

[46] Thus, in particular, for all $v_i, v_j \in \mathcal{V}$ such that $\Pi_1(v_i) = \Pi_1(v_j)$, we always have $w_i = w_j$.

[47] And hence for all $(i, j) \in \vec{\mathcal{E}}$ with $i > j$, $Y_{ij} = -Y_{ji}$.

In particular, we do not assume the ϵ shocks are uncorrelated or identically distributed across differing pairs in $\mathcal{E}$. This flexibility allows for a wide range of interpretations, ranging from models in which subjects compute compensation differences from random utilities, to models in which shocks instead emerge due to idiosyncratic, pair-specific measurement or rounding errors.

We wish to test whether the vector of population compensation differences Y^0 arises from some some additive-equivariant function U, satisfying some collection of shape constraints $\mathcal{K}$. Phrased formally, we are interested in the hypothesis:

$$H_0 : Y^0 \in \operatorname{grad}(\mathcal{K}), \qquad H_1 : Y^0 \notin \operatorname{grad}(\mathcal{K}), \tag{12}$$

where $\mathcal{K} \subseteq \mathcal{U}$ is a set of shape constraints capturing the desired properties of U.[48] Following (5), the null and alternative hypotheses in (12) can be rephrased, up to a monotone transformation, as:

$$H_0 : \min_{u \in \mathcal{K}} \left\|(\operatorname{grad} u) - Y^0\right\|_2 = 0, \qquad H_1 : \min_{u \in \mathcal{K}} \left\|(\operatorname{grad} u) - Y^0\right\|_2 > 0. \tag{13}$$

A natural sample analogue of the objective function (13) is:

$$V(\bar{Y}) = \min_{u \in \mathcal{K}} \left\|(\operatorname{grad} u) - \bar{Y}\right\|_2 = \min_{\hat{Y} \in \operatorname{grad}(\mathcal{K})} \|\hat{Y} - \bar{Y}\|_2 \tag{14}$$

where $\bar{Y}$ denotes the sample average $\frac{1}{N}\sum_n Y^n$. Under our assumptions on the data generating process, $\bar{Y}$ is a consistent estimator of Y^0, thus intuitively we should reject the null hypothesis when $V(\bar{Y})$ is large.

The numerical derivative estimator of Hong and Li (2020) provides a convenient method of obtaining critical values for the statistic (14):

1. For $b = 1, \dots, B$, let $Z^{*(b)} = \sqrt{N}\big(\bar{Y}^{*(b)} - \bar{Y}\big)$, where $\bar{Y}^{*(b)}$ is a bootstrapped sample mean, drawn from the sample $\{Y^1, \dots, Y^N\}$.[49]

2. For all $b = 1, \dots, B$, compute:

$$\hat{V}'\big(Z^{*(b)}\big) = \frac{V\big(\bar{Y} + \epsilon_N Z^{*(b)}\big) - V\big(\bar{Y}\big)}{\epsilon_N},$$

for a choice of sequence of tuning parameters ϵ_N satisfying $\lim_N \epsilon_N = 0$, and $\lim_N \epsilon_N \sqrt{N} \to \infty$.

[48] Recall from our definition of shape constraints we are guaranteed $\operatorname{grad}(\mathcal{K}) \subseteq \mathcal{F}$ is closed and convex.

[49] Note that the assumptions on our data generating process are sufficient to guarantee the consistency of the bootstrap.

3. Use the empirical distribution of $\{\hat{V}_N'\big(Z^{*(b)}\big)\}_{b=1}^{B}$ to obtain critical values for (14).

When an explicit description of the tangent cones of $\text{grad}(\mathcal{K})$ is readily available, one can modify this procedure to make use of this extra analytic information (Fang and Santos, 2019). Similarly, when $\text{grad}(K)$ is a closed, convex cone, Fang and Seo (2019) discuss a modification of this procedure with fine statistical properties.

Appendix A Proof of Theorem 1

Theorem 1. *Suppose that ϕ is a continuous action of $\mathbb{R}_+$ on X. Then a continuous preference $\succsim$ on X satisfies (N.1) - (N.3) if and only if it admits a representation by a continuous, additive-equivariant utility.*

Proof. It is immediate that if a preference relation admits a continuous additive-equivariant utility then it must satisfy (N.1) - (N.3), thus we focus on sufficiency.

Suppose then that $\succsim$ is a continuous weak order on X satisfying (N.1) - (N.3), and that there exists a $\succsim$ least-preferred alternative, $\underline{x}$. For all $x \in X$, define $c(x)$ as the (unique) solution to:

$$\phi\big(c(x), \underline{x}\big) \sim x.$$

For each x, existence of $c(x)$ follows from (N.3) and uniqueness from (N.2). Moreover, suppose $x \succsim y$. Then:

$$\phi\big(c(x), \underline{x}\big) \sim x \succsim y \sim \phi\big(c(y), \underline{x}\big),$$

hence by (N.3) there exists $\alpha \geq 0$ such that $\phi\big(\alpha, \phi(c(y), \underline{x})\big) = \phi\big(\alpha + c(y), \underline{x}\big) \sim \phi\big(c(x), \underline{x}\big)$. Thus by (N.2), $\alpha + c(y) = c(x)$, and hence $c(x) \geq c(y)$. Thus $c(\cdot)$ represents $\succsim$. As X is metric and $\succsim$ is continuous and admits the representation c, by Debreu (1964) we conclude $\succsim$ admits a continuous utility representation $u : X \to \mathbb{R}$. Suppose $(x_n) \to x$. By continuity of u, $u(x_n) \to u(x)$. But $u(x_n) = u\big(\phi(c(x_n), \underline{x})\big)$ and $u(x) = u\big(\phi(c(x), \underline{x})\big)$. As $\succsim$ satsifies (N.2), $\phi(\cdot, \underline{x})$ and $u|_{\phi(\mathbb{R}_+, \underline{x})}$ are injective, hence $\bar{u} = u|_{\phi(\mathbb{R}_+, \underline{x})} \circ \phi(\cdot, \underline{x})$ is injective and continuous. Thus as $\bar{u}\big(c(x_n)\big) \to \bar{u}\big(c(x)\big)$, $c(x_n) \to c(x)$, and as $x_n \to x$ was arbitrary, c is continuous.

To establish the additive-equivariance of c, note that by definition, for all x:

$$\phi\big(c(x), \underline{x}\big) \sim x. \tag{15}$$

Hence for all $x \in X$ and all $\alpha \geq 0$:

$$\phi\big(c(\phi(\alpha, x)), \underline{x}\big) \sim \phi(\alpha, x). \tag{16}$$

But by (15) and (N.1),

$$\phi\big(\alpha, \phi(c(x), \underline{x})\big) \sim \phi(\alpha, x), \tag{17}$$

and, as ϕ is an action:

$$\phi\big(\alpha, \phi(c(x), \underline{x})\big) = \phi\big(\alpha + c(x), \underline{x}\big). \tag{18}$$

Then by (16) - (18):

$$\phi(\alpha + c(x), \underline{x}) \sim \phi\big(c(\phi(\alpha, x)), \underline{x}\big),$$

and by (N.2) we conclude:

$$\alpha + c(x) = c(\phi(\alpha, x)). \tag{19}$$

Thus c is a continuous, additive-equivariant representation of $\succsim$.

Suppose now that $\succsim$ has no least-preferred alternative. Let $\underline{x} \in X$ be arbitrary, and define $c_{\underline{x}}(x)$ for all x in the upper contour set $\{x \in X : x \succsim \underline{x}\}$, as the unique solution to $\phi\big(c_{\underline{x}}(x), \underline{x}\big) \sim x$. By the preceding argument, $c_{\underline{x}}(\cdot)$ is continuous, additive-equivariant, and represents $\succsim$ on this subset of X. For any $x \in X$, define $c(x)$ as $c_{\underline{x}}(x)$ if $x \succsim \underline{x}$, and otherwise as $-d_x$, where d_x is the unique solution to:

$$\phi(d_x, x) \sim \underline{x}.$$

Note that such a d_x exists and is unique for each x by (N.3) and (N.2) respectively. Suppose $x \succsim y$. If $x \succsim \underline{x}$, then clearly $c(x) \geq c(y)$.[50] Consider then the case in which neither belongs to the $\underline{x}$ upper contour set. By (N.3) there exists $\alpha \geq 0$ such that:

$$\phi(\alpha, y) \sim x.$$

Then $\phi(d_x + \alpha, y) \sim \underline{x} \sim \phi(d_x, x)$, and by (N.2), $d_y = d_x + \alpha \geq d_x$, and therefore $c(x) \geq c(y)$. Thus c represents $\succsim$.

Let $\alpha \geq 0$. Since $c\big(\phi(\alpha, x)\big) = \alpha + c(x)$ if $x \succsim \underline{x}$, suppose instead $x \prec \underline{x}$. If $d_x \geq \alpha$, then:

$$\phi\big(d_x - \alpha, \phi(\alpha, x)\big) \sim \underline{x},$$

and hence $c\big(\phi(\alpha, x)\big) = -(d_x - \alpha) = c(x) + \alpha$. If, instead $\alpha > d_x$, then:

$$\phi(\alpha, x) = \phi\big(\alpha - d_x, \phi(d_x, x)\big) \sim \phi(\alpha - d_x, \underline{x}),$$

and thus $c\big(\phi(\alpha, x)\big) = \alpha - d_x + (0) = \alpha + c(x)$. Thus c is additive-equivariant.

[50] Either $y \succsim \underline{x}$ also and hence this follows from the preceding argument, or $y \prec \underline{x}$ in which case $c(x) \geq 0 > c(y)$.

Suppose now $x' \prec \underline{x}$. By hypothesis there is no $\succsim$-minimal element, hence there exists $y \in X$ such that $y \prec x' \prec \underline{x}$. Define $c_y(x)$ for all $x \succsim y$ as the unique solution to $\phi(c_y(x), y) \sim x$. By the preceding argument, c_y is continuous. Then for all $y \precsim x \prec \underline{x}$:

$$\phi\big(d_x, \phi(c_y(x), y)\big) \sim \underline{x}$$

by (N.1), thus

$$\phi\big(d_x + c_y(x), y\big) \sim \underline{x}.$$

By additive-equivariance of c_y:

$$d_x + c_y(x) + c_y(y) = c_y(\underline{x}),$$

and since $c_y(y) = 0$ by definition, re-arranging we obtain:

$$-d_x = c_y(x) - c_y(\underline{x}).$$

In particular, since $x \prec \underline{x}$, $-d_x = c(x)$. Thus:

$$c(x) = c_y(x) - c_y(\underline{x}).$$

Thus for any $y \prec \underline{x}$, the restriction of c to $\{x \in X : x \succsim y\}$ is continuous as it differs from the continuous function c_y by the constant, $c_y(\bar{x})$; hence c is continuous at x' in particular. Since every $x' \prec \underline{x}$ is contained within the upper contour set of some such y, we conclude that $c(x)$ is continuous. □

Appendix B Proof of Theorem 2

Theorem 2. *Suppose an agent has preferences $\succsim$ on X that satisfy (N.1) - (N.3), and preferences $\succsim^*$ over X^* that are consistent with $\succsim$. Then choosing to submit a bid equal to their true compensation difference, in the mechanism corresponding to the more-preferred alternative, is $\succsim^*$-optimal.*

Proof. Without loss of generality, let $x \succsim y$, with true compensation difference given by $\alpha \geq 0$, $\phi(\alpha, y) \sim x$. Since $\succsim$ satisfies (N.2) and (N.3), this α exists and is unique. Suppose first that the subject chooses to participate in the y-mechanism and submits a price of s. Then their state-dependent payoff is:

$$f_s(b_x, b_y, z) = \begin{cases} \phi(b_x, y) & \text{if } z = x \\ \phi(b_y, x) & \text{if } z = y,\ b_y \geq s \\ y & \text{if } z = y,\ s > b_y. \end{cases}$$

Similarly, if the agent instead submitted s in the x-mechanism, their reward would be:

$$g_s(b_x, b_y, z) = \begin{cases} \phi(b_y, x) & \text{if } z = y \\ \phi(b_x, y) & \text{if } z = x,\ b_x \geq s \\ x & \text{if } z = x,\ s > b_x \end{cases}$$

Suppose $s = \alpha$. By (N.2):

$$\phi(b_x, y) \succsim x \iff b_x \geq \alpha,$$

hence conditional upon $z = x$, the agent obtains $\max\{\phi(b_x, y), x\}$ from g_α.[51] Now, by (N.2), $\phi(b_y, x) \succsim y$ no matter the value of b_y, hence by consistency of $\succsim^*$ the most-preferred f act resulting from a bid in the y-mechanism is f_0.[52] Thus we wish to show $g_\alpha \succsim^* f_0$. But conditional upon $z = y$, both g_α and f_0 yield $\phi(b_y, x)$, and conditional upon $z = x$, g_α yields $\max\{\phi(b_x, y), x\}$ whereas f_0 yields $\phi(b_x, y)$. Thus by consistency, $g_\alpha \succsim^* f_0$. The final step is to show that $g_\alpha \succsim^* g_s$ for all other choices of s. This follows from the standard argument characterizing weak optimality of truthful bidding in Vickrey auctions, and we omit it. □

Appendix C Proof of Theorem 3

C.1 Overview

The proof of Theorem 3 proceeds in several steps. Firstly, we show that, even though X may itself not (up to homeomorphism) have any product structure, that if (A.1) - (A.3) hold, then there is an embedding of $X/\sim_\trianglelefteq \times \mathbb{R}_+$ into X in an equivariant fashion. Lemmas 1 to 5 verify the majority of the basic properties. Lemmas 6-8 are of a purely technical nature and together establish the continuity of the inverse of this embedding, which is only able to be defined implicitly.

We refer to the image of this embedding as $\bar{X} \subseteq X$. A rationalizable data set admits a utility representation over the finite set of alternatives featuring in the experiment; this gives a real-valued function on some finite subset $\mathcal{V}$ of $X/\sim_\trianglelefteq$. Subtracting this function from its maximal value yields a function whose graph may be interpreted as finite sample from an indifference curve of some preference. By Tietze, this function admits a bounded, continuous extension to all of $X/\sim_\trianglelefteq$, whose graph yields a single 'full' indifference curve. We then translate this curve forward and backward

[51]The max here is understood in the preference sense.

[52]That is, it comes from setting $s = 0$.

using the action ϕ to obtain a continuous weak order that both satisfies (N.1) - (N.3) and extends the observed data.

C.2 Construction of Embedding

Lemma 1. *Let ϕ be a continuous action of $\mathbb{R}_+$ on X satisfying (A.1). Define the relation $x \sim_{\trianglelefteq} y$ if either:*

$$\exists\, \alpha \geq 0 \text{ s.t. } \phi(\alpha, x) = y,$$

or

$$\exists\, \beta \geq 0 \text{ s.t. } \phi(\beta, y) = x.$$

Then $\sim_{\trianglelefteq}$ is an equivalence relation.

Proof. Clearly $\sim_{\trianglelefteq}$ is reflexive and symmetric, hence all that remains is to verify transitivity. Suppose $x \sim_{\trianglelefteq} y$ and $y \sim_{\trianglelefteq} z$. We proceed in three cases: first suppose that only one of x and z is reachable from y; without loss $x \trianglelefteq y \trianglelefteq z$. Then there exists $\alpha_{xy}, \alpha_{yz} \geq 0$ such that $\phi(\alpha_{xy}, x) = y$ and $\phi(\alpha_{yz}, y) = z$ then clearly $\phi(\alpha_{xy} + \alpha_{yz}, x) = z$ and hence $x \trianglelefteq z$. Thus suppose $y \trianglelefteq x$ and $y \trianglelefteq z$. Then there exists $\alpha_{yz}, \alpha_{yz} \geq 0$ such that $\phi(\alpha_{yx}, y) = x$ and $\phi(\alpha_{yz}, y) = z$. Without loss of generality let $\alpha_{yx} \leq \alpha_{yz}$, so:

$$\phi\big(\alpha_{yz} - \alpha_{yx}, \phi(\alpha_{yx}.y)\big) = z,$$

and thus

$$\phi(\alpha_{yz} - \alpha_{yx}, x) = z,$$

and we obtain $x \sim_{\trianglelefteq} z$. Finally, suppose $x \trianglelefteq y$ and $z \trianglelefteq y$. Then there exists $\alpha_{xy}, \alpha_{zy} \geq 0$ such that $\phi(\alpha_{xy}, x) = y = \phi(\alpha_{zy}, z)$. Without loss, let $\alpha_{xy} \leq \alpha_{zy}$. Then:

$$\begin{aligned} y &= \phi(\alpha_{zy}, z) \\ &= \phi(\alpha_{xy} + (\alpha_{zy} - \alpha_{xy}), z) \\ &= \phi\big(\alpha_{xy}, \phi(\alpha_{zy} - \alpha_{xy}, z)\big). \end{aligned}$$

But, by (A.1), $\phi(\alpha_{xy}, \cdot)$ is injective hence, $\phi(\alpha_{zy} - \alpha_{xy}, z) = x$ and therefore $x \sim_{\trianglelefteq} z$. $\square$

In light of Lemma 1, there is a well-defined quotient space $X/{\sim_{\trianglelefteq}}$. In all that follows, we will consider $X/{\sim_{\trianglelefteq}}$ endowed with its quotient topology.

Corollary 1. *Let $q : X \to X/\sim_{\trianglelefteq}$ denote the canonical quotient map. Then for all $\alpha \geq 0$, for all $x \in X$,*

$$q(x) = (q \circ \phi)(\alpha, x).$$

Lemma 2. *Suppose (A.1) and (A.2). Then any continuous cross section s is an embedding of $X/\sim_{\trianglelefteq}$ into X.*

Proof. By hypothesis, s is continuous. Suppose then that $s(y') = s(y)$ for $y, y' \in X/\sim_{\trianglelefteq}$. Then:

$$(q \circ s)(y') = (q \circ s)(y)$$

and hence $y = y'$ as s is a cross section; thus s is injective. Moreover, by hypothesis, $q|_{\text{range}(s)} : \text{range}(s) \to X/\sim_{\trianglelefteq}$ is an inverse and continuous as $X/\sim_{\trianglelefteq}$ carries the quotient topology. Hence s is open. □

For some fixed cross section s, define $\bar{s} : \mathbb{R}_+ \times X/\sim_{\trianglelefteq} \to X$ via:

$$\bar{s}(\alpha, y) = \phi\big(\alpha, s(y)\big),$$

and let $\bar{X} = \text{range}(\bar{s})$. We wish to show that $\bar{s}$ is an equivariant embedding, where the $\mathbb{R}_+$ acts on the domain by addition along the first factor. Clearly equivariance holds by construction:

$$\begin{aligned}
\phi\big(\beta, \bar{s}(\alpha, y)\big) &= \phi\big(\beta, \phi(\alpha, s(y))\big) \\
&= \phi\big(\beta + \alpha, s(y)\big) \\
&= \bar{s}(\beta + \alpha, y).
\end{aligned}$$

In all that follows we will assume (A.1) and (A.2), and a fixed s and hence fixed $\bar{s}$.

Lemma 3. *Let $\bar{q} : \bar{X} \to X/\sim_{\trianglelefteq}$ be the restriction of q to $\bar{X}$. Then $\bar{q}$ is an open map.*

Proof. Let $U \subset \bar{X}$ be open. Then:

$$\begin{aligned}
\bar{q}(U) &= \big\{y \in X/\sim_{\trianglelefteq} : \exists\, \alpha \geq 0 \text{ s.t. } \phi\big(\alpha, s(y)\big) \in U\big\} \\
&= s^{-1}\big(\{x \in \text{range}(s) : \exists\, \alpha \geq 0 \text{ s.t. } \phi(\alpha, x) \in U\}\big) \\
&= s^{-1}\big(\text{range}(s) \cap \big[\cup_{\alpha \geq 0} f_\alpha^{-1}(U)\big]\big),
\end{aligned}$$

where $f_\alpha = \phi(\alpha, \cdot)$. But, for all $\alpha \geq 0$, $\phi(\alpha, \cdot)$ is continuous hence $\text{range}(s) \cap \big[\cup_{\alpha \geq 0} f_\alpha^{-1}(U)\big]$ is a relatively open subset of $\text{range}(s)$. Hence by Lemma 2, $\bar{q}(U)$ is open. □

Lemma 4. *Suppose that, for all $x \in X$, $\phi(\cdot, x)$ is injective. Then $\bar{s}$ is injective.*

Proof. Suppose $\bar{s}(\alpha, y) = \bar{s}(\alpha', y')$. Then:

$$\begin{aligned} \phi\big(\alpha, s(y)\big) &= \phi\big(\alpha', s(y')\big) \\ (q \circ \phi)\big(\alpha, s(y)\big) &= (q \circ \phi)\big(\alpha', s(y')\big) \\ s(y) &= s(y') \\ y &= y' \end{aligned}$$

where the second-to-last equality follows from Corollary 1, and the last from invoking Lemma 2. As $\phi\big(\cdot, s(y)\big)$ is injective, $\alpha = \alpha'$, and hence $\bar{s}$ is injective. □

For the remainder of this section, we will assume $\phi(\cdot, x)$ is injective for all x. Define $t : \bar{X} \to \mathbb{R}_+$ pointwise as the unique solution to:

$$\phi\big(t(x), (s \circ \bar{q})(x)\big) = x.$$

We will first show that the map $(t, \bar{q})$ is indeed the inverse of $\bar{s}$ (Lemma 5). We then establish the regularity (i.e. continuity) of solutions to the above class of topological implicit function problems (Lemma 6 - Lemma 8).

Lemma 5. *The map $(t, \bar{q}) : \bar{X} \to \mathbb{R}_+ \times X/{\sim_\trianglelefteq}$ is the inverse of $\bar{s}$.*

Proof. We will show $(t, \bar{q})$ is a left inverse. Thus let $(\alpha, y) \in \mathbb{R}_+ \times X/{\sim_\trianglelefteq}$. Then:

$$\begin{aligned} \big((t, \bar{q}) \circ \bar{s}\big)(\alpha, y) &= \big((t \circ \bar{s})(\alpha, y), (\bar{q} \circ \bar{s})(\alpha, y)\big) \\ &= \big((t \circ \bar{s})(\alpha, y), (q \circ \phi)(\alpha, s(y))\big) \\ &= \big((t \circ \bar{s})(\alpha, y), y\big), \end{aligned}$$

where the final equality follows from Corollary 1. Hence it remains to show $(t \circ \bar{s})(\alpha, y) = \alpha$. By definition of t,

$$\phi\big((t \circ \bar{s})(\alpha, y), (s \circ \bar{q} \circ \bar{s})(\alpha, y)\big) = \bar{s}(\alpha, y),$$

but by plugging in for $\bar{s}$ and appeal to Corollary 1, this simplifies to:

$$\phi\big((t \circ \bar{s})(\alpha, y), s(y)\big) = \phi\big(\alpha, s(y)\big).$$

Since $\phi\big(\cdot, s(y)\big)$ is injective, this implies $(t \circ \bar{s})(\alpha, y) = \alpha$ as desired. □

Lemma 6. *Suppose (A.1) - (A.3) and that ϕ is injective in its first factor. Then, for all $x \in \bar{X}$ there exists a finite open cover $\{N_{\alpha_i}\}_{i=1}^K$ of $\bar{s}\big([0, t(x)] \times \{\bar{q}(x)\}\big)$ with the following properties:*

1. *For all $i \in \{1, \ldots, K\}$, the set $\{\alpha : \bar{s}(\alpha, \bar{q}(x)) \in N_{\alpha_i}\}$ is a (relatively) open interval of $[0, \infty)$. For $i > 1$, denote this by $(\underline{\alpha}_i, \bar{\alpha}_i)$, and for $i = 1$, by $[0, \bar{\alpha}_1)$.*

2. *The indices $\{\alpha_i\}_{i=1}^K$ satisfy $0 = \alpha_1 < \alpha_2 < \cdots < \alpha_K = t(x)$, satisfy $\alpha_i \in (\underline{\alpha}_i, \bar{\alpha}_i)$, and, for all $i, j = 1, \ldots, K$, $\alpha_i < \alpha_j$ implies $(\underline{\alpha}_i, \bar{\alpha}_i) \preceq_{SSO} (\underline{\alpha}_j, \bar{\alpha}_j)$, where $\prec_{SSO}$ denotes the strong set order.*

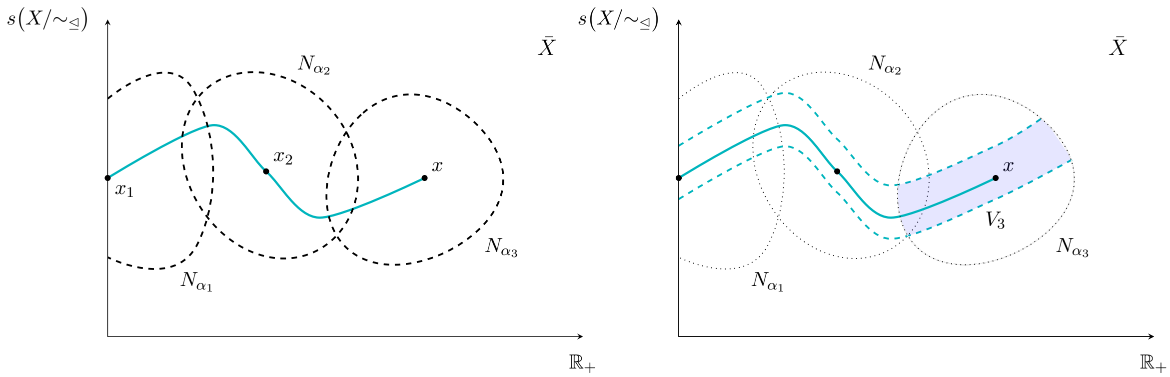
3. *For all i, N_{α_i} satisfies the no loitering property of (A.3).*

Proof. Fix $x \in \bar{X}$. For all $\alpha \in [0, t(x)]$, define $x_\alpha = \bar{s}(\alpha, \bar{q}(x)) = \phi\big(\alpha, (s \circ \bar{q})(x)\big)$. By (A.3), for all $\alpha \in [0, t(x)]$, there exists $\varepsilon_\alpha, T_\alpha > 0$ such that, for all $x' \in B_{\varepsilon_\alpha}(x_\alpha)$, for all $\beta > T_\alpha$, $\phi(\beta, x') \notin B_{\varepsilon_\alpha}(x_\alpha)$. For each α, let U_α denote the connected component of $B_{\varepsilon_\alpha}(x_\alpha) \cap \bar{s}\big([0, t(x)] \times \{\bar{q}(x)\}\big)$ that contains x_α, and define $N_\alpha = B_{\varepsilon_\alpha}(x_\alpha) \setminus \big[\bar{s}\big([0, t(x)] \times \{\bar{q}(x)\}\big) \setminus U_\alpha\big]$. As $[0, t(x)] \times \{\bar{q}(x)\}$ is compact in $\mathbb{R}_+ \times X/\sim_{\trianglelefteq}$, by continuity $\bar{s}\big([0, t(x)] \times \{\bar{q}(x)\}\big)$ is a compact and hence closed subset of $\bar{X}$. U_α is a relatively open subset of $\bar{s}\big([0, t(x)] \times \{\bar{q}(x)\}\big)$, hence $\bar{s}\big([0, t(x)] \times \{\bar{q}(x)\}\big) \setminus U_\alpha$ is relatively closed in $\bar{s}\big([0, t(x)] \times \{\bar{q}(x)\}\big)$ and therefore also closed in $\bar{X}$. Then for all α, N_α is an open neighborhood of x_α. Moreover, by Lemma 4, $\bar{s}\big(\cdot, \bar{q}(x)\big)$ is injective (and continuous) hence for all α, $\{\alpha' : \bar{s}\big(\alpha', \bar{q}(x)\big) \in N_\alpha\}$ is an open interval in $[0, t(x)]$.

As $\bar{s}\big([0, t(x)] \times \{\bar{q}(x)\}\big)$ is compact and covered by $\{N_\alpha\}_{\alpha \in [0, t(x)]}$, there exists a finite set $0 = \alpha_1 < \cdots < \alpha_K = t(x)$ such that $\{N_{\alpha_i}\}_{i=1}^K$ form a finite subcover. By construction, for each i, $\alpha_i \in (\underline{\alpha}_i, \bar{\alpha}_i)$. Moreover, since properties (1.) and (3.) held for every element of $\{N_\alpha\}$ they hold for $\{N_{\alpha_i}\}$. Finally, it is without loss of generality to suppose that for all $i \neq j$, the intervals $(\underline{\alpha}_i, \bar{\alpha}_i) \not\subseteq (\underline{\alpha}_j, \bar{\alpha}_j)$, as if not, then some proper subcover does, and passing to this subcover preserves properties (1.) and (3.).

Then it remains only to verify $\{N_{\alpha_i}\}$ has the property that $\alpha_i < \alpha_j$ implies $(\underline{\alpha}_i, \bar{\alpha}_i) \preceq_{SSO} (\underline{\alpha}_j, \bar{\alpha}_j)$. Since neither interval contains the other, if $\underline{\alpha}_i < \underline{\alpha}_j$, then it must be that $\bar{\alpha}_i < \bar{\alpha}_j$, which implies $(\underline{\alpha}_i, \bar{\alpha}_i) \preceq_{SSO} (\underline{\alpha}_j, \bar{\alpha}_j)$ as desired.[53] If instead $\underline{\alpha}_j < \underline{\alpha}_i$, then $\bar{\alpha}_j < \bar{\alpha}_i$, in which case $(\underline{\alpha}_j, \bar{\alpha}_j) \preceq_{SSO} (\underline{\alpha}_i, \bar{\alpha}_i)$, and hence $\alpha_i, \alpha_j \in (\underline{\alpha}_i, \bar{\alpha}_i) \cap (\underline{\alpha}_j, \bar{\alpha}_j)$. Thus swapping the labels of N_{α_i} and N_{α_j} preserves all salient properties but 'fixes' violations of property (2.). Repeating this process

[53] Note that as no interval in the collection is a subset of any other, it can never be the case that $\underline{\alpha}_i = \underline{\alpha}_j$ or $\bar{\alpha}_i = \bar{\alpha}_j$, thus considering only strict inequalities suffices.



(a) An open cover of the path $\bar{s}\big([0,t(x)] \times \{\bar{q}(x)\}\big)$, here in aquamarine. This open cover satisfies all of the properties of Lemma 6.

(b) The construction of the neighborhood V_K (here, $K = 3$) for x on which t is bounded, from the open cover $\{N_{\alpha_i}\}_{i=1}^3$.

Figure 3: An illustration of the construction underpinning Lemma 7. We have implicitly drawn the numeraire-paths of ϕ in $\bar{X}$ as vertical translates of one another.

for each such pair cannot cycle (it simply sorts the indices via the $\{\underline{\alpha}_i\}$) and thus it terminates after some finite number of label swaps, resulting in a cover satisfying (2.). $\square$

Lemma 7. *Suppose (A.1) - (A.3) and that ϕ is injective in its first factor. Then for all $x \in \bar{X}$ there exists some open neighborhood of x on which t is bounded.*

Proof. Fix $x \in \bar{X}$, and let $\{N_{\alpha_i}\}_{i=1}^K$ denote an open cover of $\bar{s}\big([0,t(x)] \times \{\bar{q}(x)\}\big)$ of the form guaranteed by Lemma 6. Without loss of generality, suppose that N_{α_1} is the sole element to intersect $\bar{s}\big(\{0\} \times X/\sim_{\trianglelefteq}\big)$.[54] Define:

$$V_0 = \bar{s}\big(\{0\} \times X/\sim_{\trianglelefteq}\big)$$

and, for all $i = 1, \ldots, K$:

$$V_i = N_{\alpha_i} \bigcap \left[(\bar{q}^{-1} \circ \bar{q}) \left(\bigcup_{j<i} V_j \cap N_{\alpha_i} \right) \right],$$

see Figure 3. We first verify, for all $i = 1, \ldots, K$, that V_i is open. Note that via Lemma 3 and our assumption that N_{α_1} is the only element of the open cover to intersect V_0, it suffices to show that

[54] For example, for all $i > 1$, redefine $N'_{\alpha_i} = N_{\alpha_i} \setminus \mathrm{range}(s)$. N'_{α_i} is open as $\mathrm{range}(s)$ is closed: let $(x_n) \in \mathrm{range}(s)$ and suppose $x_n \to x$. Then $q(x_n) \to q(x)$, and hence $(s \circ q)(x_n) \to (s \circ q)(x)$ by continuity. However, s is a cross-section thus, as $x_n \in \mathrm{range}(s)$, x_n must be the value s takes at $q(x_n)$, hence $(s \circ q)(x_n) = x_n$ for all n. As X is metric and hence Hausdorff and as x_n converges to both x and $(s \circ q)(x)$, $(s \circ q)(x)$ must equal x, and thus $x \in \mathrm{range}(s)$.

V_1 is open. But

$$V_1 = N_{\alpha_1} \cap \big(\bar{q}^{-1} \circ \bar{q}\big)(V_0 \cap N_{\alpha_1}),$$

and $V_0 \cap N_{\alpha_1} = N_{\alpha_1} \cap \text{range}(s)$, and hence is relatively open in the range of s. As $\bar{q}$ is a left-inverse of s, $\bar{q}(N_{\alpha_1} \cap V_0)$ is open, and hence so too is V_1.

We now establish that, for all $i = 1, \ldots, K$,

$$\bar{s}\big([0, \bar{\alpha}_i) \times \{\bar{q}(x)\}\big) \subseteq \bigcup_{j \leq i} V_j,$$

where we recall that $(\underline{\alpha}_i, \bar{\alpha}_i) = \{\alpha \in [0, t(x)] : \bar{s}(\alpha, \bar{q}(x)) \in N_{\alpha_i}\}$ for $1 < i < K$, and $[0, \bar{\alpha}_i)$ is the analogue for $i = 1$.[55] For all $i = 1, \ldots, K$, let $x_{\alpha_i} = \bar{s}(\alpha_i, \bar{q}(x))$ and consider the case of $i = 1$. By hypothesis, $\alpha_1 = 0$, hence $x_{\alpha_1} = (s \circ \bar{q})(x) \in N_{\alpha_1} \cap V_0$. Then Corollary 1 implies $\bar{s}\big([0, t(x)] \times \{\bar{q}(x)\}\big) \subseteq (\bar{q}^{-1} \circ \bar{q})(N_{\alpha_1} \cap V_0)$, and thus $\bar{s}\big([0, \bar{\alpha}_1) \times \{\bar{q}(x)\}\big) \subseteq V_1$. Suppose now that, for all $1 \leq i \leq k$, that:

$$\bar{s}\big([0, \bar{\alpha}_i) \times \{\bar{q}(x)\}\big) \subseteq \bigcup_{j \leq i} V_j,$$

but, for sake of contradiction, suppose that:

$$\bar{s}\big([0, \bar{\alpha}_{k+1}) \times \{\bar{q}(x)\}\big) \not\subseteq \bigcup_{j \leq k+1} V_j.$$

As $(\underline{\alpha}_{k+1}, \bar{\alpha}_{k+1})$ is an interval, if $\bar{\alpha}_k \in (\underline{\alpha}_{k+1}, \bar{\alpha}_{k+1})$, the contradiction hypothesis would be false, thus it must be that $\bar{\alpha}_k \notin (\underline{\alpha}_{k+1}, \bar{\alpha}_{k+1})$ and hence $\bar{s}(\bar{\alpha}_k, \bar{q}(x)) \notin N_{\alpha_{k+1}}$. Then $(\underline{\alpha}_k, \bar{\alpha}_k) \cap (\underline{\alpha}_{k+1}, \bar{\alpha}_{k+1}) = \varnothing$. But Lemma 6 guarantees that, for all $l > k+1$, $\underline{\alpha}_l > \underline{\alpha}_{k+1}$, and for all $l < k$, $\bar{\alpha}_l < \bar{\alpha}_k$, hence:

$$\bar{s}(\bar{\alpha}_k, \bar{q}(x)) \notin \bigcup_{i=1}^{K} N_{\alpha_i},$$

contradicting the fact that $\{N_{\alpha_i}\}_{i=1}^{K}$ is a cover for $\bar{s}\big([0, t(x)] \times \{\bar{q}(x)\}\big)$. Thus by induction $\bar{s}\big([0, \bar{\alpha}_K) \times \{\bar{q}(x)\}\big) \subseteq \bigcup_{j \leq K} V_j$, and in particular $x = x_{\alpha_K} \in V_K$.

We now verify that $t|_{V_i}$ is bounded for all $i = 0, \ldots, K$; since $x \in V_K$ and V_K is open, this suffices to establish the claim. For $i = 0$ the claim is trivial as by definition, $t|_{V_0}$ is uniformly 0.

[55] This set is indeed an interval by Lemma 6.

Thus consider $i = 1$, let $x' \in V_1$. Note that for any $\underline{x}' \in V_1$, if $\phi(\alpha, \underline{x}') = x'$, then $t(x') = \alpha + t(\underline{x}')$ by equivariance of $\bar{s}$.[56] But since N_{α_1} has a no-loitering bound of T_{α_1}, since both $x', \underline{x}' \in V_1 \subseteq N_{\alpha_1}$,

$$t(x') < T_{\alpha_1} + t(\underline{x}').$$

However, if $x' \in V_1$, then $(s \circ \bar{q})(x') \in V_1$, and by definition $(t \circ s \circ \bar{q})(x') = 0$. Thus for all $x' \in V_1$, $t(x') < T_{\alpha_1}$. Suppose now that, for all $i \leq k$, $t|_{V_i}$ is bounded, and let $x' \in V_{k+1}$. Then, $x' \in N_{\alpha_{k+1}}$ and there exists some $x'' \sim_{\trianglelefteq} x'$, where $x'' \in N_{\alpha_i} \cap V_j$ where $1 \leq j \leq k$. Suppose $x'' \trianglelefteq x'$. Then:

$$\begin{aligned} t(x') &< t(x'') + T_{\alpha_{k+1}} \\ &< \bar{T}_j + T_{\alpha_{k+1}} \\ &\leq \max_{i \leq k} \bar{T}_i + T_{\alpha_{k+1}}, \end{aligned}$$

where $T_{\alpha_{k+1}}$ is a no-loitering bound for $N_{\alpha_{k+1}}$, and $\bar{T}_j$ is any upper bound on $t|_{V_j}$ which exists by the induction hypothesis. Note that if $x' \trianglelefteq x''$, then $t(x')$ is bounded above by the same quantity. Thus for all $1 \leq i \leq K$, $t|_{V_i}$ is bounded; as $x \in V_K$ and V_K is open, this establishes the claim. □

Lemma 8. *Suppose (A.1) - (A.3) and that ϕ is injective in its first factor. Then t is continuous.*

Proof. Fix $x \in \bar{X}$. By Lemma 7, there exists $\varepsilon > 0$ such that $t|_{B_\varepsilon(x)}$ is bounded above by some constant K. Define $t^* : B_\varepsilon(x) \rightrightarrows \mathbb{R}_+$ via:

$$t^*(x') = \underset{\tilde{t} \in [0,K]}{\arg\min}\, d_X\big(\phi(\tilde{t}, (s \circ \bar{q})(x')), x'\big).$$

Since $t(x')$ is the unique unconstrained minimizer of this objective function, and $t(x') \in [0, K]$, it follows that $t^* = t|_{B_\varepsilon(x)}$ and hence t^* is a singleton-valued correspondence. But by the Theorem of the Maximum (Aliprantis and Border, 2006), t^* is upper hemicontinuous and hence continuous as a function. Thus for every $x \in X$ there is a restriction of t to some neighborhood of x on which it is continuous, hence it is continuous. □

[56] By definition,

$$\phi\big(t(\underline{x}'), (s \circ \bar{q})(\underline{x}')\big) = \underline{x}'$$

and, appealing to Corollary 1 to conclude $(s \circ \bar{q} \circ \phi)(\alpha, \underline{x}') = (s \circ \bar{q})(\underline{x}')$,

$$\phi\big((t \circ \phi)(\alpha, \underline{x}'), (s \circ \bar{q})(\underline{x}')\big) = \phi(\alpha, \underline{x}').$$

But as ϕ is an action:

$$\phi\big(\alpha + t(\underline{x}'), (s \circ \bar{q})(\underline{x}')\big) = \phi(\alpha, x')$$

too. By injectivity of ϕ in its first component, we conclude $t(\underline{x}') + \alpha = (t \circ \phi)(\alpha, \underline{x}') = t(x')$.

Corollary 2. *Suppose (A.1)-(A.3), and that $\phi(\cdot, x)$ is injective for all $x \in X$. Then $\bar{s}$ is an equivariant embedding.*

Theorem 3. *Let (X, ϕ) satisfy (A.1) - (A.3) and suppose Π_ϕ is non-empty. Then for every experiment $\mathcal{E}$, for any dataset, the following are equivalent:*

(i) The data are cardinally consistent.

(ii) The data are rationalized by a continuous preference that satisfies (N.1) - (N.3).

(iii) The data are rationalized by a continuous, additive-equivariant utility function.

Proof. (i) $\Longrightarrow$ (ii): Let $\mathcal{E}$ be an experiment, and $F \in \mathcal{F}$ a cardinally consistent flow on $(\mathcal{V}, \mathcal{E})$. By Proposition 1, there exists a utility $u \in \mathcal{U}$ such that $\operatorname{grad} u = F$. Without loss of generality we may suppose u is non-negative valued (by adding an appropriate constant that has no effect on its gradient). Define $u'(v) = u_{\max} - u(v)$, where $u_{\max}$ is the greatest value u assumes on $\mathcal{V}$. Since we have assumed in the definition of an experiment that, for all pairs (intra observation or across observations), no two elements of $\mathcal{V}$ are related under $\sim_{\trianglelefteq}$, then $q(\mathcal{V})$ is in one-to-one correspondence with $\mathcal{V}$ in the natural manner. Thus $u' : \mathcal{V} \to \mathbb{R}_+$ may be equivalently regarded as a map $\tilde{u}' : q(\mathcal{V}) \to \mathbb{R}_+$. By Lemma 2, $X/{\sim_{\trianglelefteq}}$ is homeomorphic to a subset of X and hence is metrizable and thus normal. Therefore, by the Tietze extension theorem, e.g. Munkres (1974), there exists a bounded, continuous function $U : X/{\sim_{\trianglelefteq}} \to \mathbb{R}_+$ such that $U|_{q(\mathcal{V})} = \tilde{u}'$.

We define a binary relation on X in three stages: first, if $x, y \in \bar{s}\big(\operatorname{epi}(U)\big) \subseteq \bar{X}$, then let $x \succsim y$ if and only if $t(x) - t(y) \geq (U \circ \bar{q})(x) - (U \circ \bar{q})(y)$. Note that here $t(x)$ and $t(y)$ are well defined because $x, y \in \bar{X}$. If x but not y belong to $\bar{s}\big(\operatorname{epi}(U)\big)$, then let $x \succ y$. Finally, if neither x nor y belong to $\bar{s}\big(\operatorname{epi}(U)\big)$, then let:

$$x \succsim y \iff \min\{\alpha \in \mathbb{R}_+ : \phi(\alpha, y) \in \bar{s}\big(\operatorname{epi}(U)\big)\} \geq \min\{\alpha \in \mathbb{R}_+ : \phi(\alpha, x) \in \bar{s}\big(\operatorname{epi}(U)\big)\}.$$

Note that both minima are taken over closed sets that are bounded below and hence exist, the inequality on the right-hand side is well-defined. As these cases are exhaustive, $\succsim$ is complete. Now let $x \succsim y$ and $y \succsim z$, and suppose first that $x, y, z \in \bar{s}\big(\operatorname{epi}(U)\big)$. Then

$$t(x) - t(y) \geq (U \circ \bar{q})(x) - (U \circ \bar{q})(y),$$

and

$$t(y) - t(z) \geq (U \circ \bar{q})(y) - (U \circ \bar{q})(z),$$

hence summing one obtains $t(x) - t(z) \geq (U \circ \bar{q})(x) - (U \circ \bar{q})(z)$ and thus $x \succsim z$. Clearly, if $x, y \in \bar{s}\big(\text{epi}(U)\big)$ but z is not, then $x \succsim z$, and by definition it is impossible that $y, z \in \bar{s}\big(\text{epi}(U)\big)$ but x is not, as $x \succsim y$. Thus suppose now that $x, y, z \notin \bar{s}\big(\text{epi}(U)\big)$. But then $x \succsim z$ by virtue of $\geq$ being a transitive order on $\mathbb{R}$. Thus $\succsim$ is transitive and hence a preference relation.

We now establish that $\succsim$ is continuous. First, let $x \in \bar{s}\big(\text{epi}(U)\big)$. Then, noting that $y \succsim x$ only if $y \in \bar{s}\big(\text{epi}(U)\big)$:

$$\begin{aligned}\{y \in X : y \succsim x\} &= \{y \in \bar{X} : t(y) - t(x) \geq (U \circ \bar{q})(y) - (U \circ \bar{q})(x)\},\\ &= \{y \in \bar{X} : t(y) - (U \circ \bar{q})(y) \geq t(x) - (U \circ \bar{q})(x)\},\end{aligned}$$

where we define $\delta_x \equiv t(x) - (U \circ \bar{q})(x) \geq 0$. Consider the function $U_x : X/{\sim_{\trianglelefteq}} \to \mathbb{R}$ where $U_x(y) = U(y) + \delta_x$. This is continuous as U is and, by definition, $\{y \in X : y \succsim x\} = \bar{s}\big(\text{epi}(U_x)\big)$. By Corollary 2, this set is closed as $\text{epi}(U_x)$ is. Similarly, $\{y \in X : y \succ x\} = \bar{s}\big(\text{int}\,\text{epi}(U_x)\big)$, hence it is open; as $\succsim$ is complete, $\{y \in X : y \precsim x\} = \bar{s}\big(\text{int}\,\text{epi}(U_x)\big)^c$ is closed.

Suppose now that $x \notin \bar{s}\big(\text{epi}(U)\big)$, and let $\hat{\alpha}_x = \min\{\alpha \in \mathbb{R}_+ : \phi(\alpha, x) \in \bar{s}\big(\text{epi}(U)\big)\}$. Then $\phi(\hat{\alpha}_x, x) = \bar{s}\big((U \circ q)(x), q(x)\big)$, hence:

$$\begin{aligned}\{y \in X : y \succsim x\} &= \{y \in \bar{s}\big(\text{epi}(U)\big)^c : y \succsim x\} \cup \bar{s}\big(\text{epi}(U)\big)\\ &= \{y \in \bar{s}\big(\text{epi}(U)\big)^c : \hat{\alpha}_x \geq \hat{\alpha}_y\} \cup \bar{s}\big(\text{epi}(U)\big)\\ &= \{y \in \bar{s}\big(\text{epi}(U)\big)^c : y \in f^{-1}_{\hat{\alpha}_x}\big(\bar{s}\big(\text{epi}(U)\big)\big)\} \cup \bar{s}\big(\text{epi}(U)\big)\\ &= f^{-1}_{\hat{\alpha}_x}\big(\bar{s}\big(\text{epi}(U)\big)\big).\end{aligned}$$

where $f_{\hat{\alpha}_x} = \phi(\hat{\alpha}_x, \cdot)$. As $f_{\hat{\alpha}_x}$ is continuous and $\bar{s}\big(\text{epi}(U)\big)$ is closed, we conclude the weak upper contour set at x is closed. Analogously, the strict upper contour set at x is open, and therefore the weak lower contour set at x is closed too. As these cases are exhaustive, $\succsim$ is continuous.

We now verify $\succsim$ obeys (N.1) - (N.3). Suppose then that $x \succsim y$, and let $\alpha \geq 0$. If $x, y \in$

$\bar{s}\big(\text{epi}(U)\big)$, then, as $\phi(\alpha, x) = \phi\big(\alpha, \phi(t(x), (s \circ q)(x))\big)$ (and likewise y):

$$\begin{aligned}
(t \circ \phi)(\alpha, x) - (t \circ \phi)(\alpha, y) &= (t \circ \phi)\big(\alpha, \phi(t(x), (s \circ q)(x))\big) - (t \circ \phi)\big(\alpha, \phi(t(x), (s \circ q)(x))\big) \\
&= (t \circ \phi)\big(\alpha + t(x), (s \circ q)(x)\big) - (t \circ \phi)\big(\alpha + t(y), (s \circ q)(y)\big) \\
&= (t \circ \bar{s})(\alpha + t(x), q(x)) - (t \circ \bar{s})(\alpha + t(y), q(y)) \\
&= (\alpha + t(x)) - (\alpha + t(y)) \\
&= t(x) - t(y) \\
&\geq (U \circ \bar{q})(x) - (U \circ \bar{q})(y) \\
&= (U \circ \bar{q} \circ \phi)(\alpha, x) - (U \circ \bar{q} \circ \phi)(\alpha, y)
\end{aligned}$$

where the inequality follows from $x \succsim y$ and $x, y \in \bar{s}\big(\text{epi}(U)\big)$. Thus $\phi(\alpha, x) \succsim \phi(\alpha, y)$, as $\phi(\alpha, x), \phi(\alpha, y) \in \bar{s}\big(\text{epi}(U)\big)$. Suppose now x but not y belongs to $\bar{s}\big(\text{epi}(U)\big)$ (and thus that $x \succ y$. Then for all $0 \leq \alpha < \hat{\alpha}_y$, by definition $\phi(\alpha, x) \succ \phi(\alpha, y)$, hence suppose $\alpha \geq \hat{\alpha}_y$. Then as shown above, $(t \circ \phi)(\alpha, x) = t(x) + \alpha$, where $t(x) \geq (U \circ q)(x)$. Similarly, since $y \notin \bar{s}\big(\text{epi}(U)\big)$, $(t \circ \phi)(\alpha, y) < (U \circ q)(y) + \alpha$. Hence:

$$\begin{aligned}
(t \circ \phi)(\alpha, x) - (t \circ \phi)(\alpha, y) &= t(x) + \alpha - (t \circ \phi)(\alpha, y) \\
&\geq (U \circ q)(x) + \alpha - (t \circ \phi)(\alpha, y) \\
&> (U \circ q)(x) - (U \circ q)(y) \\
&= (U \circ q \circ \phi)(\alpha, x) - (U \circ q \circ \phi)(\alpha, y),
\end{aligned}$$

hence $\phi(\alpha, x) \succ \phi(\alpha, y)$. Finally, suppose neither x nor y belong to $\bar{s}\big(\text{epi}(U)\big)$. Let $x \succsim y$ hence $\hat{\alpha}_y \geq \hat{\alpha}_x$. For all $\alpha < \hat{\alpha}_x$, $\hat{\alpha}_{\phi(\alpha,x)} = \hat{\alpha}_x - \alpha$, thus for all such α, $\phi(\alpha, x) \succsim \phi(\alpha, y)$. If $\alpha \geq \hat{\alpha}_x$, then $\phi(\alpha, x) \in \bar{s}\big(\text{epi}(U)\big)$; if $\phi(\alpha, y)$ is not then the preceding argument implies $\phi(\alpha, x) \succ \phi(\alpha, y)$. If $\phi(\alpha, y) \in \bar{s}\big(\text{epi}(U)\big)$, then:

$$\begin{aligned}
(t \circ \phi)(\alpha, x) - (t \circ \phi)(\alpha, y) &= \hat{\alpha}_y - \hat{\alpha}_x \\
&\geq (U \circ q \circ \phi)(\alpha, x) - (U \circ q \circ \phi)(\alpha, y).
\end{aligned}$$

Thus $\succsim$ satsifies (N.1). Property (N.2) holds by definition. Thus now suppose $y \succsim x$. Then $\phi(\hat{\alpha}_x, x), \phi(\hat{\alpha}_x, y) \in \bar{s}\big(\text{epi}(U)\big)$, thus, having verified (N.1) it suffices to find some α such that:

$$\phi(\alpha + \hat{\alpha}_x, x) \sim \phi(\hat{\alpha}_x, y).$$

Let:

$$\alpha = \big((t \circ \phi)(\hat{\alpha}_x, y) - (U \circ q)(y)\big).$$

Note this is well-defined as $\phi(\hat{\alpha}_x, y) \in \bar{s}\big(\mathrm{epi}(U)\big)$. But, since $(t \circ \phi)(\hat{\alpha}_x, x) = (U \circ q)(x)$,

$$\begin{aligned}(t \circ \phi)(\alpha + \hat{\alpha}_x, x) - (t \circ \phi)(\hat{\alpha}_x, y) &= \alpha + (t \circ \phi)(\hat{\alpha}_x, x) - (t \circ \phi)(\hat{\alpha}_x, y) \\ &= \alpha + (U \circ q)(x) - (t \circ \phi)(\hat{\alpha}_x, y) \\ &= (U \circ q)(x) - (U \circ q)(y).\end{aligned}$$

Thus $\succsim$ satisfies (N.3), our last outstanding claim. Thus (i) $\implies$ (ii).

That (ii) $\implies$ (iii) follows from Theorem 1, so it remains only to prove (iii) $\implies$ (i). Let $x_0, \dots, x_L \in X$, and suppose that $U : X \to \mathbb{R}$ is additive-equivariant. Clearly:

$$\sum_{l=0}^{L} U(x_{l+1}) - U(x_l) = 0,$$

where subscripts are understood mod-L. Let:

$$\alpha_l = \begin{cases} \alpha_{l,l+1} & \text{if } x_{l+1} \sim \phi(\alpha_{l,l+1}, x_l) \\ -\alpha_{l,l+1} & \text{if } x_l \sim \phi(\alpha_{l,l+1}, x_{l+1}). \end{cases}$$

Then, for all $l = 0, \dots L$, if $U(x_{l+1}) \geq U(x_l)$:

$$U(x_{l+1}) - U(x_l) = U(x_l) + \alpha_{l,l+1} - U(x_l) = \alpha_l,$$

and if $U(x_l) \geq U(x_{l+1})$:

$$U(x_{l+1}) - U(x_l) = U(x_{l+1}) - \big[U(x_{l+1}) + \alpha_{l,l+1}\big] = \alpha_l.$$

Thus:

$$0 = \sum_{l=0}^{L} U(x_{l+1}) - U(x_l) = \sum_{l=0}^{L} \alpha_l.$$

Thus the compensation differences arising from any additive-equivariant utility will always be cardinally consistent. $\square$

Remark 5. Conditions (A.2) - (A.3) are also necessary in the following sense. Suppose X is a metric space, ϕ a continuous action of $\mathbb{R}_+$ on X, and that (A.1) holds so that (A.2) is well-defined. Then if there exists an equivariant embedding $\hat{s} : \mathbb{R}_+ \times X/\!\sim_{\trianglelefteq} \to X$ (where the action of $\mathbb{R}_+$ on $\mathbb{R}_+ \times X/\!\sim_{\trianglelefteq}$ is simply addition along the first factor), then (A.2) and (A.3) must hold. This suggests that the technical conditions of Theorem 3 cannot be significantly relaxed without requiring a completely different proof approach.

Appendix D Proposition Proofs

D.1 Proof of Proposition 1

Proof. Suppose first that $F \in \text{im}(\text{grad})$. Then there exists $u \in \mathcal{U}$ such that $\text{grad}\, u = F$. Let $(v^0, v^1), (v^1, v^2), \ldots, (v^L, v^0) \in \vec{\mathcal{E}}$. Then:

$$\sum_{l=0}^{L} F_{v^l v^{l+1}} = \sum_{l=0}^{L} (\text{grad}\, u)_{v^l v^{l+1}} = \sum_{l=0}^{L} (u_{v^{l+1}} - u_{v^l}) = 0.$$

Conversely, suppose F is cardinally consistent. Let $(\mathcal{V}, \mathcal{E}')$ denote a spanning tree for $(\mathcal{V}, \mathcal{E})$. Fix $\underline{v} \in \mathcal{V}$. Then for each $v \neq \underline{v}$, there is a unique sequence of edges in $\vec{\mathcal{E}}'$:

$$(\underline{v}, v^1), (v^1, v^2), \ldots, (v^L, v)$$

connecting $\underline{v}$ to v. Define $u(\underline{v}) = 0$ and:

$$u(v) = F_{\underline{v} v^1} + \sum_{l=1}^{L-1} F_{v^l v^{l+1}} + F_{v^L v}$$

The utility u is well-defined and does not depend on the choice of spanning tree: this follows from observing that if, for two different choices of spanning tree, the sums of F along two different paths from $\underline{v}$ to v differed, then by reversing one of the paths, one would obtain a violation of cardinal consistency. Finally, by construction, $\text{grad}\, u = F$, completing the proof. □

D.2 Proof of Proposition 2

Proof. By the rank-nullity theorem, it suffices to verify that, for all $u \in \mathcal{U}$, $F \in \mathcal{F}$:

$$\langle -\text{div}\, F, u \rangle = \langle F, \text{grad}\, u \rangle,$$

where $\mathcal{U}$ carries its standard Euclidean inner product. Then:

$$\begin{aligned}
\langle F, \operatorname{grad} u\rangle &= \sum_{\{(i,j)\in\vec{\mathcal{E}}\,:\,i<j\}} F_{ij}\left[u_j - u_i\right] \\
&= \sum_{\{(i,j)\in\vec{\mathcal{E}}\,:\,i<j\}} F_{ij}u_j + \sum_{\{(i,j)\in\vec{\mathcal{E}}\,:\,i<j\}} F_{ji}u_i \\
&= \sum_{\{(i,j)\in\vec{\mathcal{E}}\,:\,j<i\}} F_{ji}u_i + \sum_{\{(i,j)\in\vec{\mathcal{E}}\,:\,i<j\}} F_{ji}u_i \\
&= \sum_{(i,j)\in\vec{\mathcal{E}}} F_{ji}u_i \\
&= \sum_{i\in\mathcal{V}}\left[\sum_{j\in N(i)} F_{ji}\right]u_i \\
&= \sum_{i\in\mathcal{V}}\left[-\sum_{j\in N(i)} F_{ij}\right]u_i \\
&= \langle -\text{div } F, u\rangle,
\end{aligned}$$

where the third to last line follows from the observation that summing over $\vec{\mathcal{E}}$ (i.e. each edge twice, once with each orientation) is equivalent to summing over, for each v_i, all of the edges connecting v_i to its neighbors, oriented away from v_i. Thus $\mathcal{F}$ admits an orthogonal decomposition as $\text{im}(\text{grad}) \oplus \ker(\text{div})$. □

D.3 Proof of Proposition 3

Proof. As noted in the text, for a pure cycle C, $MP(C) = \|C\|_1$. Thus if $R = \sum_l C_l$ for some $\{C_1, \ldots, C_L\} \in \mathfrak{D}(R)$, then by the triangle inequality:

$$\|R\|_1 = \left\|\sum_{l=1}^{L} C_l\right\|_1 \leq \sum_{l=1}^{L}\left\|C_l\right\|_1 = \sum_{l=1}^{L} MP(C_l).$$

Taking infimums across all such decompositions of R we obtain $\|R\|_1 \leq MP^*(R)$. Thus it suffices to show that there always exists a decomposition in $\mathfrak{D}(R)$ attaining this lower bound.

Without loss of generality, suppose $R \geq 0$ componentwise.[57] If $R = 0$ then trivially $MP^*(R) = \|R\|_1 = 0$, hence suppose $R \neq 0$. Let $\mathcal{E}'$ denote the subset of edges on which $R \neq 0$, and let

[57] This simply amounts to a choice of orientation of each edge forming our basis for $\mathcal{F}$ in the same direction as the flow (if the flow is non-zero).

$\mathcal{V}' = \cup_{\{x,y\}\in\mathcal{E}'}\{x,y\}$ denote the associated vertex set. Choose $v^0 \in \mathcal{V}'$ arbitrarily. Since $v^0 \in \mathcal{V}'$ and $R \neq 0$, there exists some $v^1 \in N(v_0)$ such that $R_{v^0 v^1} \neq 0$. Since $R \in \ker(\mathrm{div})$, v_1 may be chosen so that $R_{v^0 v^1} > 0$. Proceeding analogously we may construct a sequence of oriented edges in $\vec{\mathcal{E}}'$ such that $R_{v^j v^{j+1}} > 0$. We terminate this process when we choose a vertex that has appeared prior in the sequence.[58] Possibly by throwing out some initial segment of this sequence and relabelling indices, we obtain a sequence of oriented edges $(v^0, v^1), (v^1, v^2), \dots, (v^{J_1}, v^0)$ such that $R_{v^j v^{j+1}} > 0$. Let $c_1 = \min_j R_{v^j v^{j+1}}$, and let $C_1 = \sum_{j=0}^{J_1} c_1 \mathbb{1}_{(v^j, v^j+1)}$. Then $0 \leq C_1 \leq R$ component-wise, and C_1 is equal to R on at least one component. Thus $R^1 = R - C_1$ also belongs to the positive cone of $\ker(\mathrm{div})$; however it is supported on a strict subgraph of $(\mathcal{V}', \mathcal{E}')$. Thus repeating this process, we obtain a finite decomposition $R = C_1 + \cdots + C_L$, where for all l, $C_l \geq 0$. Since every $C_l \geq 0$, however:

$$\|R\|_1 = \left\| \sum_l^L C_l \right\|_1 = \sum_{l=1}^L \left\|C_l\right\|_1 = \sum_{l=1}^L MP(C_l)$$

and hence the lower bound obtains. □

Appendix E Shape Constraint 'Cookbook'

E.1 Quaslinear, Increasing, Concave Utility (Proof of Example 7)

Proof. Suppose first that U is a quasilinear, increasing, and concave utility. For all $i = 1, \dots, K$, define $u_i = U(v_i)$ and let π_i denote an arbitrary choice of supergradient of U at each v_i. As U is increasing, it follows $\pi_i \geq 0$ for each i. Define $\gamma_i = u_i - \langle \pi_i, v_i \rangle$. Then for all $i = 1, \dots, K$ and all $x \in X$:

$$U(x) \leq U(v_i) + \langle \pi_i, x - v_i \rangle.$$

Thus, in particular, $\langle \pi_i, v_i \rangle + \gamma_i \leq \langle \pi_j, v_i \rangle + \gamma_j$ for all i, j. Finally, as:

$$U\big(\phi(\alpha, v_i)\big) \leq U(v_i) + \langle \pi_i, (\alpha, 0) \rangle$$

it follows that:

$$\alpha \leq \pi_i^1 \alpha$$

hence $\pi^1 \geq 1$. If v_i is on the interior of $\mathbb{R}_+^2$ then there is some $\hat{v}$ such that, for some $\hat{\alpha} > 0$, $\phi(\hat{\alpha}, \hat{v}) = v_i$. Thus $U(\hat{v}) = U(v_i) - \alpha$, and:

$$U(\hat{v}) \leq U(v_i) + \langle \pi_i, (-\alpha, 0) \rangle,$$

[58] This process necessarily termiantes as $(\mathcal{V}, \mathcal{E})$ is finite.

which yields $-\alpha \leq -\alpha \pi_i^1$ and hence $\pi_i^1 \leq 1$. Thus for all v in the interior of X, their supergradients must have first component equal to 1. By the outer hemicontinuity of the supergradient correspondence (Hiriart-Urruty and Lemaréchal (2004), Theorem 6.2.4) this remains true for those v on the boundary of X, and hence for all v_i, π_i is of the form $(1, \pi_i^2)$ as claimed.

Conversely, suppose $u, \{\pi_i\}_{i=1}^K, \{\gamma_i\}_{i=1}^K$ is a solution to (6). Define:

$$\tilde{U}(x) = \min_{i \in \{1,\ldots,K\}} \gamma_i + \langle x, \pi_i \rangle.$$

Then clearly $U(v_i) = u_i$, and $\tilde{U}$ is quasilinear, increasing, and concave. □

E.2 Cobb-Douglas Preferences (Proof of Example 8)

Proof. Consider $X = \mathbb{R}_{++}^L$, and $\phi(\alpha, x) = e^\alpha x$. Define $H : X \to \mathbb{R}^L$ via:

$$H(x) = \big(\ln x_1, \ldots, \ln x_L\big).$$

The transformation H induces an action of $\mathbb{R}_+$ on $\mathbb{R}^L$ via $\tilde{\phi}(\alpha, H(x)) = H(\phi(\alpha, x))$, here given by:

$$\tilde{\phi}(\alpha, H(x)) = H(x) + \alpha \mathbb{1}_L.$$

Critically, (X, ϕ) and $(\mathbb{R}^L, \tilde{\phi})$ are isomorphic in the above sense, and hence there is a one-to-one correspondence between observations of the form:

$$\phi(\alpha, x) \sim y$$

with:

$$\tilde{\phi}(\alpha, H(x)) \sim H(y).$$

A collection of observations of this latter form is rationalized by an affine utility on $H(X)$ (with gradient in $\Delta(L)$) if and only if the former form is rationalized by a Cobb-Douglas utility, hence (5) under change of coordinates becomes:

$$\begin{aligned} \min_{u \in \mathcal{U}} \quad & \big\|(\operatorname{grad} u) - \bar{Y}\big\|_2^2 \\ \text{subject to} \quad & u_i = \big\langle \beta, H(v_i) \big\rangle \quad \forall i = 1, \ldots, K \\ & \langle \beta, \mathbb{1}_L \rangle = 1 \\ & \beta \geq 0 \end{aligned} \tag{20}$$

for $\beta \in \mathbb{R}^L$. □

E.3 Risk-Neutral Utility Functionals on $\mathbb{R}^S$

Let S be a finite set of states of the world and let $X = \mathbb{R}^S$ denote the space of monetary acts, along with $\phi(\alpha, x) = x + \alpha \mathbb{1}_S$. Let $(\mathcal{V}, \mathcal{E})$ denote an experiment; recall by definition, there does not exist any pair $v_i, v_j \in \mathcal{V}$ such for which there is some $\alpha \geq 0$ such that $\phi(\alpha, v_i) = v_j$. In light of Remark 4, we will drop the 'risk-neutral' qualifier as it is understood that these characterizations may be straightforwardly extended to other Bernoulli utilities.

E.3.1 Subjective Expected Utility

A map $U : X \to \mathbb{R}$ is said to be a subjective expected utility functional if it is of the form:

$$U(x) = \langle \pi, x \rangle,$$

for some $\pi \in \Delta(S)$. Define $\mathcal{K}_{\text{SEU}}$ as the collection of $u \in \mathcal{U}$ that are restrictions of subjective expected utility functionals. Then solving (5) with $\mathcal{K} = \mathcal{K}_{\text{SEU}}$ is equivalent to solving:

$$\begin{aligned} \min_{u \in \mathcal{U}} \quad & \left\| (\operatorname{grad} u) - \bar{Y} \right\|_2^2 \\ \text{subject to} \quad & u_i = \langle \pi, v_i \rangle \qquad \forall i = 1, \ldots, K \\ & \langle \pi, \mathbb{1}_S \rangle = 1 \\ & \pi \geq 0. \end{aligned} \tag{21}$$

Proof. Trivial. □

E.3.2 Choquet Expected Utility

Recall that a function $\nu : 2^S \to \mathbb{R}$ is a capacity if (i) $\nu(\varnothing) = 0$, $\nu(S) = 1$, and (ii) for all $A \subseteq B$, $\nu(A) \leq \nu(B)$. By abuse of notation, let $S = \{1, \ldots, S\}$, and let $\mathfrak{S}_S$ denote the set of permutations on $\{1, \ldots, S\}$. For each $\sigma \in \mathfrak{S}_S$, define:

$$C_\sigma = \{x \in \mathbb{R}^S : x_{\sigma(1)} \geq x_{\sigma(2)} \geq \cdots \geq x_{\sigma(S)}\}. \tag{22}$$

The cones $\{C_\sigma\}_{\sigma \in \mathfrak{S}_S}$ cover $\mathbb{R}^S$. Note that if a functional $U : \mathbb{R}^S \to \mathbb{R}$ corresponds to Choquet integration with respect to ν, then for any σ, $U|_{C_\sigma}$ is linear, and indeed if $x \in C_\sigma$, then:

$$U(x) = \int_S x \, dP^\sigma,$$

where, for all $i = 1, \ldots, S$, the probability measure P^σ is defined by:

$$P^\sigma(\sigma(i)) = \nu\big(\{\sigma(1), \sigma(2), \ldots, \sigma(i)\}\big) - \nu\big(\{\sigma(1), \sigma(2), \ldots, \sigma(i-1)\}\big). \tag{23}$$

See Ghirardato et al. (2004) for more discussion. Finally, for notational simplicity, define the shorthand A_i^σ for the set $\{\sigma(1), \sigma(2), \ldots, \sigma(i)\}$.

We say that $U : X \to \mathbb{R}$ is said to be a Choquet expected utility (CEU) functional if:

$$U(x) = \int_S x \, d\nu,$$

where ν is a capacity the integral denotes Choquet integration. Define $\mathcal{K}_{CEU}$ as the collection of $u \in \mathcal{U}$ that are restrictions of CEU functionals. Then solving (5) with $\mathcal{K} = \mathcal{K}_{\text{CEU}}$ is equivalent to solving:

$$\begin{aligned}
\min_{u \in \mathcal{U}} \quad & \left\| (\operatorname{grad} u) - \bar{Y} \right\|_2^2 \\
\text{subject to} \quad & u_i = \langle P^\sigma, v_i \rangle && \forall \sigma \in \mathfrak{S}_S, \forall i = 1, \ldots, K \text{ s.t. } v_i \in C^\sigma \\
& P^\sigma_{\sigma(j)} = \nu_{A_j^\sigma} - \nu_{A_{j-1}^\sigma} && \forall \sigma \in \mathfrak{S}_S, \forall j = 1, \ldots, S \\
& \nu_A \leq \nu_B && \forall A, B \in 2^S \text{ s.t. } A \subseteq B \\
& \nu_\varnothing = 0 \\
& \nu_S = 1
\end{aligned} \tag{24}$$

Proof. Suppose U is a CEU functional. Then it corresponds to integration against some capacity ν which by definition then satisfies the last three constraints of (24). From the discussion, e.g., in Ghirardato et al. (2004) (see, in particular, Example 17), each v_i belongs to at least one C_σ cone, and restricted to each, U simply amounts to integration (i.e. a dot product) of v_i with the measure P^σ. Hence every CEU functional corresponds to a solution to (24).

Conversely, suppose u, ν, and the corresponding $\{P^\sigma\}_{\sigma \in \mathfrak{S}_S}$ constitute a solution to (24). Defining $U(x) = \langle P^\sigma, x \rangle$, where P^σ corresponds to the measure associated with any σ for which $x \in C_\sigma$ yields an extension of u corresponding to Choquet integration against ν. □

E.3.3 Convex Choquet Expected Utility

A capacity $\nu : 2^S \to \mathbb{R}$ is said to be a convex, if, for all $A, B \subseteq S$:

$$\nu(A) + \nu(B) \leq \nu(A \cap B) + \nu(A \cup B).$$

A map $U : X \to \mathbb{R}$ is said to be a convex Choquet expected utility (CCEU) functional if it is of the form:

$$U(x) = \int_S x \, d\nu,$$

for some convex capacity ν. Define $\mathcal{K}_{CCEU}$ as the collection of $u \in \mathcal{U}$ that are restrictions of CCEU functionals. Then, solving (5) with $\mathcal{K} = \mathcal{K}_{\mathrm{CCEU}}$ is equivalent to solving:

$$\begin{aligned}
\min_{u \in \mathcal{U}} \quad & \left\|(\operatorname{grad} u) - \bar{Y}\right\|_2^2 \\
\text{subject to} \quad & u_i = \langle P^\sigma, v_i \rangle && \forall \sigma \in \mathfrak{S}_S, \forall i = 1, \ldots, K \text{ s.t. } v_i \in C^\sigma \\
& P^\sigma_{\sigma(j)} = \nu_{A^\sigma_j} - \nu_{A^\sigma_{j-1}} && \forall \sigma \in \mathfrak{S}_S, \forall j = 1, \ldots, S \\
& \nu_A \leq \nu_B && \forall A, B \in 2^S \text{ s.t. } A \subseteq B \\
& \nu_A + \nu_B \leq \nu_{A \cup B} + \nu_{A \cap B} && \forall A, B \in 2^S \\
& \nu_\varnothing = 0 \\
& \nu_S = 1
\end{aligned} \tag{25}$$

Proof. Follows from CEU case, where additionally the supermodularity of ν is enforced. □

E.3.4 Maxmin Expected Utility

A map $U : X \to \mathbb{R}$ is said to be a maxmin expected utility (MEU) functional if it is of the form:

$$U(x) = \min_{\pi \in P} \langle \pi, x \rangle,$$

for some compact, convex belief set $P \subseteq \Delta(S)$. Define $\mathcal{K}_{\mathrm{MEU}}$ as the collection of $u \in \mathcal{U}$ that are restrictions of MEU functionals. Then solving (5) with $\mathcal{K} = \mathcal{K}_{\mathrm{MEU}}$ is equivalent to solving:

$$\begin{aligned}
\min_{u \in \mathcal{U}} \quad & \left\|(\operatorname{grad} u) - \bar{Y}\right\|_2^2 \\
\text{subject to} \quad & u_i = \langle \pi_i, v_i \rangle && \forall i = 1, \ldots, K \\
& \langle \pi_i, v_i \rangle \leq \langle \pi_j, v_i \rangle && \forall i, j = 1, \ldots, K \\
& \langle \pi_i, \mathbb{1}_S \rangle = 1 && \forall i = 1, \ldots, K \\
& \pi_i \geq 0 && \forall i = 1, \ldots, K,
\end{aligned} \tag{26}$$

for $\pi_1 \ldots, \pi_K \in \mathbb{R}^S$.

Proof. Suppose first that $u \in \mathcal{K}$ is the restriction to $\mathcal{V}$ of some MEU functional U. For $i = 1, \ldots, K$, let $\pi_i \in \partial U(v_i)$ denote an arbitrarily selection of supergradients of U. As $U(0) = 0$, by homogeneity, $U(v_i) = \langle \pi_i, v_i \rangle$ for all $i = 1, \ldots, K$. Furthermore, for all $x \in \mathbb{R}^S$ and all $v_i \in \mathcal{V}$:

$$\begin{aligned} U(x) &\leq U(v_i) + \langle \pi_i, x - v_i \rangle \\ &= \langle \pi_i, v_i \rangle + \langle \pi_i, x - v_i \rangle \\ &= \langle \pi_i, x \rangle, \end{aligned}$$

hence for all $v_j \in \mathcal{V}$, $\langle \pi_j, v_j \rangle \leq \langle \pi_i, v_j \rangle$. As U is increasing, for each i, $\pi_i \geq 0$. Let $\alpha \in \mathbb{R}$. Since U is translation-invariant, for all v_i:

$$U(v_i + \alpha \mathbb{1}_S) \leq U(v_i) + \langle \pi_i, \alpha \mathbb{1}_S \rangle$$

hence

$$U(v_i) + \alpha \leq U(v_i) + \langle \pi_i, \alpha \mathbb{1}_S \rangle$$

and

$$\alpha \leq \alpha \langle \pi_i, \mathbb{1}_S \rangle. \tag{27}$$

If $\alpha > 0$, $1 \leq \langle \pi, \mathbb{1}_S \rangle$, and if $\alpha < 0$, $1 \geq \langle \pi, \mathbb{1}_S \rangle$. Since (28) holds for all $\alpha \in \mathbb{R}$, we obtain $\langle \pi_i, \mathbb{1}_S \rangle = 1$.

Suppose now that for some collection $\pi_1, \ldots, \pi_K \in \Delta(S)$, we have a vector $u \in \mathcal{U}$ satisfying (i) $u_i = \langle \pi_i, v_i \rangle$ and (ii) $\langle \pi_i, v_i \rangle \leq \langle \pi_j, v_i \rangle$. Define

$$\hat{U}(x) = \min_{i \in \{1, \ldots, K\}} \langle \pi_i, x \rangle = \min_{\pi \in \mathrm{co}\{\pi_1, \ldots, \pi_K\}} \langle \pi, x \rangle.$$

The latter equality follows from standard results on support functions see, e.g., Hiriart-Urruty and Lemaréchal (2004) Theorem 3.3.2. By construction, $u_i = \hat{U}(v_i)$ and $\hat{U}$ is a risk-neutral MEU functional. $\square$

E.3.5 Variational Preferences (Proof of Example 9)

Proof. Suppose first that $u \in \mathcal{K}$ is the restriction to $\mathcal{V}$ of some risk-neutral variational utility functional U. For $i = 1, \ldots, K$, let $\pi_i \in \partial U(v_i)$ be an arbitrary selection of supergradients of U, one at each v_i. For all $i = 1, \ldots K$, let:

$$\gamma_i = u_i - \langle \pi_i, v_i \rangle.$$

Then, for all i, by construction $u_i = \gamma_i + \langle \pi_i, v_i \rangle$ and $\gamma_K = 0$. Moreover, for all $x \in \mathbb{R}^S$ and all $v_j \in \mathcal{V}$:

$$\begin{aligned} U(x) &\leq U(v_j) + \langle \pi_j, x - v_j \rangle \\ &= \gamma_j + \langle \pi_j, v_j \rangle + \langle \pi_j, x - v_j \rangle \\ &= \gamma_j + \langle \pi_j, x \rangle, \end{aligned}$$

hence in particular, for all $v_i \in \mathcal{V}$, $\gamma_i + \langle \pi_i, v_i \rangle \leq \gamma_j + \langle \pi_j, v_i \rangle$. As U is increasing, for each i, $\pi_i \geq 0$. Let $\alpha \in \mathbb{R}$. Since U is translation-invariant, for all v_i:

$$U(v_i + \alpha \mathbb{1}_S) \leq U(v_i) + \langle \pi_i, \alpha \mathbb{1}_S \rangle$$

hence

$$U(v_i) + \alpha \leq U(v_i) + \langle \pi_i, \alpha \mathbb{1}_S \rangle$$

and

$$\alpha \leq \alpha \langle \pi_i, \mathbb{1}_S \rangle. \tag{28}$$

If $\alpha > 0$, $1 \leq \langle \pi, \mathbb{1}_S \rangle$, and if $\alpha < 0$, $1 \geq \langle \pi, \mathbb{1}_S \rangle$. Since (28) holds for all $\alpha \in \mathbb{R}$, we obtain $\langle \pi_i, \mathbb{1}_S \rangle = 1$.

Suppose now that for some collection $\pi_1, \ldots, \pi_K \in \Delta(S)$ and $\gamma_1, \ldots, \gamma_K \in \mathbb{R}$ with $\gamma_K = 0$, we have a vector $u \in \mathcal{U}$ satisfying (i) $u_i = \gamma_i + \langle \pi_i, v_i \rangle$, and (ii) $\gamma_i + \langle \pi_i, v_i \rangle \leq \gamma_j + \langle \pi_j, v_i \rangle$. Define

$$\hat{U}(x) = \min_{i \in \{1, \ldots, K\}} \gamma_i + \langle \pi_i, x \rangle$$

By construction, $u_i = \hat{U}(v_i)$ and $\hat{U}$ is a (i) translation invariant, (ii) concave, (iii) increasing, (iv) normalized functional hence, by the results of Maccheroni et al. (2006), corresponds to a variational utility functional. □

E.3.6 Dual Self Expected Utility

A map $U : X \to \mathbb{R}$ is said to be a dual-self utility functional if it is of the form:

$$U(x) = \max_{P \in \mathbb{P}^*} \min_{\pi \in P} \langle \pi, x \rangle,$$

where $\mathbb{P}^*$ is a compact collection (in the Hausdorff topology) of compact, convex subsets of $\Delta(S)$.

Let $(\mathcal{V}, \mathcal{E})$ denote an experiment, where $v_K = 0$. Let $\mathcal{K}_{\text{DS}}$ denote the collection of $u \in \mathcal{U}$ that are restrictions of dual-self utility functionals. Then solving (5) with $\mathcal{K} = \mathcal{K}_{\text{DS}}$ is equivalent to solving:

$$\begin{aligned}
\min_{u \in \mathcal{U}} \quad & \left\|(\operatorname{grad} u) - \bar{Y}\right\|_2^2 \\
\text{subject to} \quad & u_i = \langle \pi_{ii}, v_i \rangle && \forall i = 1, \ldots, K \\
& \langle \pi_{ii}, v_i \rangle \leq \langle \pi_{ij}, v_i \rangle && \forall i, j = 1, \ldots, K \\
& \langle \pi_{ji}, v_i \rangle \leq \langle \pi_{ii}, v_i \rangle && \forall i, j = 1, \ldots, K \\
& \langle \pi_{ij}, \mathbb{1}_S \rangle = 1 && \forall i, j = 1, \ldots, K \\
& \pi_{ij} \geq 0 && \forall i, j = 1, \ldots, K,
\end{aligned} \tag{29}$$

for $u, \{\pi_{ij}\}_{i,j=1}^K \in \mathbb{R}^K$.

Proof. Suppose, first, that $u, \{\pi_{ij}\}_{i,j=1}^K$ is a solution to (29). Define, for each $i = 1, \ldots, K$, the set $P_i = \text{co}\{\pi_{i,1}, \ldots, \pi_{i,K}\}$. Clearly $P_i \subseteq \Delta(S)$ for each i. Let $\mathbb{P}^* = \{P_i\}_{i=1}^K$. We claim that:

$$U(x) = \max_{P \in \mathbb{P}^*} \min_{\pi \in P} \langle \pi, x \rangle$$

defines a DSEU functional whose restriction to $\mathcal{V}$ is precisely u. Firstly, as $\langle \pi_{ii}, v_i \rangle \leq \langle \pi_{ij}, v_i \rangle$ for all $j = 1, \ldots, K$, it follows that:

$$u_i = \langle \pi_{ii}, v_i \rangle = \min_{\pi \in P_i} \langle \pi, v_i \rangle.$$

But, for all $j = 1, \ldots, K$ we have $\langle \pi_{ji}, v_i \rangle \leq u_i$, hence:

$$u_i \geq \langle \pi_{ji}, v_i \rangle \geq \min_{\pi \in P_j} \langle \pi, v_i \rangle,$$

as $\pi_{ji} \in P_j$. Thus:

$$\begin{aligned}
U(v_i) &= \max_{P \in \mathbb{P}^*} \min_{\pi \in P} \langle \pi, v_i \rangle \\
&= \min_{\pi \in P_i} \langle \pi, v_i \rangle \\
&= \langle \pi_{ii}, v_i \rangle \\
&= u_i.
\end{aligned}$$

Suppose now that $U(x) = \max_{P \in \mathbb{P}^*} \min_{\pi \in P} \langle \pi, x \rangle$ is a DSEU functional on $\mathbb{R}^S$. For $i = 1, \dots, K$, let $P_i \in \mathbb{P}^*$ denote any belief set for which:

$$U(v_i) = \min_{\pi \in P_i} \langle \pi, v_i \rangle,$$

and let $\pi_{ii} \in P_i$ be any minimizer of the right-hand side.[59] Define, for each $i = 1, \dots, K$, the utility value $u_i = \langle \pi_{ii}, v_i \rangle$. Since P_j is an 'active' belief set at v_j for each $j \neq i$, there exists, for each j, some $\pi_{ij} \in P_i$ such that $\langle \pi_{ij}, v_j \rangle \leq u_j$. Since each $\pi_{ij} \in P_i$, then $u_i \leq \langle \pi_{ij}, v_i \rangle$ for each i. Then, as clearly every $\pi_{ij} \in \Delta(S)$, the collection $u, \{\pi_{ij}\}_{i,j=1}^K$ is a solution to (29), as required.

□

E.3.7 Dual-Self Variational Utility

A map $U : X \to \mathbb{R}$ is said to be a dual-self variational utility functional if it is of the form:

$$U(x) = \max_{c \in \mathbb{C}} \min_{\pi \in \Delta(S)} \langle \pi, x \rangle + c(\pi),$$

where $\mathbb{C}$ is a collection of convex cost functions $c : \Delta(S) \to [0, \infty]$ such that $\max_{c \in \mathbb{C}} \min_{\pi \in \Delta(S)} c(\pi) = 0$. Such functionals are characterized by being (i) additive-equivariant, (ii) monotone, (iii) normalized, i.e. $U(\mathbb{1}_S) = 1$, see Supplementary Appendix to Chandrasekher et al. (2020).

Let $(\mathcal{V}, \mathcal{E})$ denote an experiment, where $v_K = 0$. Let $\mathcal{K}_{\text{DSV}}$ denote the collection of $u \in \mathcal{U}$ that are restrictions of dual-self variational utility functionals. Then solving (5) with $\mathcal{K} = \mathcal{K}_{\text{DSV}}$ is equivalent to solving:

$$\begin{aligned} \min_{u \in \mathcal{U}} \quad & \left\| (\operatorname{grad} u) - \bar{Y} \right\|_2^2 \\ \text{subject to} \quad & u_i \geq u_j \qquad\qquad \forall i, j \text{ s.t. } v_i \geq v_j \\ & u_K = 0, \end{aligned} \tag{30}$$

where $v_i \geq v_j$ is understood in the product order on $\mathbb{R}^S$.

Proof. Firstly, suppose U is a dual-self variational functional. Then it clearly is monotone, hence $v_i \geq v_j$ implies $U(v_i) \geq U(v_j)$. Moreover,

$$U(\mathbb{1}_S) = U\big(\phi(1, 0)\big) = U(0) + 1,$$

hence U is normalized if and only if $U(0) = 0$. Thus clearly letting $u_i = U(v_i)$ satisfies the constraints of (30).

[59] Such a belief set exists as $\mathbb{P}^*$ is compact (in the Hausdorff topology on the space of compact subsets of $\Delta(S)$), and $\min_{\pi \in P} \langle \pi, x \rangle$ is continuous in P for each x.

Conversely, suppose u is a solution to (30). In light of the characterization provided in Chandrasekher et al. (2020), it suffices to prove there exists an additive-equivariant and monotone extension from $\mathcal{V}$ to $\mathbb{R}^S$.[60] However, note that since $\mathcal{V}$ contains no pairs of $\sim_{\trianglelefteq}$-related elements, u is trivially additive-equivariant and by definition monotone on $\mathcal{V}$. Hence by Theorem 1 of Cerreia-Vioglio et al. (2014),

$$U(x) = \sup\{u_{v_i} + b : v_i \in \mathcal{V}, b \in \mathbb{R}, \text{ and } v_i + b\mathbb{1}_S \leq x\}$$

defines an additive-equivariant, monotone, and normalized extension of u, and hence by Chandrasekher et al. (2020) this corresponds to some dual-self variational utility functional. □

E.4 Miscellaneous Properties

E.4.1 Additive Separability (Proof of Example 10)

Proof. Suppose u, w^1, w^2 satisfy the constraints of (11). Since ϕ acts exclusively on X_1, by Theorem 3, there exists an additive-equivariant extension of w^1 to $W^1 : X_1 \to \mathbb{R}$. Similarly, as $V_2 \subseteq X_2$ is finite, and X_2 is metric by hypothesis, by the Tietze extension theorem (e.g. Munkres 1974), there exists a continuous extension W^2 of w^2. Define:

$$\tilde{U}(x_1, x_2) = W^1(x_1) + W^2(x_2).$$

Since $W^1(v_i^1) = w_i^1$ and $W^2(v_i^2) = w_i^2$, clearly $U(v_i) = w_i^1 + w_i^2 = u_i$, and hence any such solution corresponds to an additively separable and additve-equivariant utility.

Conversely, suppose U is additive-equivariant and is given by:

$$U(x_1, x_2) = W^1(x_1) + W^2(x_2).$$

Then, letting $u_i = U(v_i)$, $w^1 = W^1(v_i^1)$, and $w_i^2 = W^2(v_i)$, u, w^1, w^2 constitute a solution to (11). □

Appendix F Miscellaneous (Counter-)examples

Example 11 (An Action With No Cross Section)**.** Let $f : \mathbb{R} \to \mathbb{R}$ be a discontinuous solution to Cauchy's functional equation:

$$f(x + y) = f(x) + f(y).$$

[60] Normalization holds for any additive-equivariant extension, as $u_K = 0$.

The graph of any such f is dense in the plane and hence is unbounded over every nontrivial interval.[61] Let $X = \{(x, y) \in \mathbb{R}^2 : y \geq f(x)\}$ be the epigraph of f, and let $\mathbb{R}_+$ act on X by addition along the second coordinate. Then $X/{\sim_{\trianglelefteq}}$ may be identified with the horizontal axis of $\mathbb{R}^2$, and a cross section simply corresponds to a continuous $s : \mathbb{R} \to \mathbb{R}$ that pointwise dominates f. However, no such s exists as any such function is bounded on compacta.

Example 12 (Additive-equivariant Representation of Von Neumann-Morgenstern Preferences)**.** Let $\tilde{X}$ be a finite, linearly ordered set with maximal element $\bar{x}$, and let $X = \Delta(\tilde{X}) \setminus \{\delta_{\bar{x}}\}$, where $\Delta(\tilde{X})$ denotes the probability simplex. Suppose $\succsim$ is a monotone (with respect to the order on $\tilde{X}$) von Neumann-Morgenstern preference on $\Delta(\tilde{X})$ hence admits a utility function:

$$\tilde{U}(p) = \sum_{x \in \tilde{X}} p_x \tilde{u}(x)$$

for all $p \in \Delta(\tilde{X})$. Without loss of generality, we suppose $\tilde{U}$ is normalized so $\tilde{U}(\delta_{\bar{x}}) = 0$. As in Example 5, we consider the restriction $\succsim \big|_X$. In particular, for all $p \in X$, this implies $U(p) < 0$. Recall we define ϕ here via:

$$\phi(\alpha, p) = e^{-\alpha} p + (1 - e^{-\alpha})\delta_{\bar{x}}.$$

Then:

$$\begin{aligned} \tilde{U}\big(\phi(\alpha, p)\big) &= e^{-\alpha}\tilde{U}(p) + (1 - e^{-\alpha})U(\delta_{\bar{x}}) \\ &= e^{-\alpha}\tilde{U}(p) \end{aligned}$$

Let $f : (-\infty, 0) \to \mathbb{R}$ via $f(y) = -\ln(-y)$, and define $U = f \circ \tilde{U}$. Note that f is strictly increasing. Then:

$$\begin{aligned} U\big(\phi(\alpha, p)\big) &= -\ln\big(- e^{-\alpha}\tilde{U}(p)\big) \\ &= -\ln\big(- \tilde{U}(p)\big) - \ln e^{-\alpha} \\ &= U(p) + \alpha. \end{aligned}$$

[61] See, for example, Aczél and Dhombres (1989) Chapter 1 Theorem 3.

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