Revealed Invariant Preference

Pete Caradonna[1] Christopher Chambers[2]

[1]Division of the Humanities and Social Sciences
California Institute of Technology

[2]Department of Economics
Georgetown University

Introduction

Nearly all economic models built on foundation of economic actors maximizing individual well-being.

- Requires specifying how actors evaluate various stylized trade-offs and decisions.
- If these assumptions inconsistent with broad empirical regularities, models can yield unrealistic/outright incorrect predictions (e.g. Mehra & Prescott '85).

Revealed Preference

A Basic Question: How do we *systematically* obtain the testable implications of various models of preference and decision?

Classically, revealed preference has studied:

(i) Testable implications of rational behavior (generally)

- → Model-free approach
- → Doesn't speak to specific structure(s) we're often interested in

(ii) Testable implications of specific theories on model-by-model basis

- → Relies on special model-specific structure; no unified theory.
- → Often relies on Afriat-type machinery; only valid for particular environments.

'Universal' Revealed Preference Theory

We are interested in studying the general mapping:

$$\text{Model} \mapsto \text{Testable Implications.}$$

A Less Basic Question: Can we obtain general results which characterize empirical content of *any* theory whose axioms belong to certain broad classes?

$\longrightarrow$ Need to exploit common mathematical structure behind various classes of axioms.

Categorizing Axioms

Rationality

- '$\succeq$ is complete and transitive.'

Monotonicity

- For some order $\trianglerighteq$ on alternatives: '$x \trianglerighteq y$ implies $x \succeq y$'
- Local non-satiation

Continuity

- Continuity, mixture-continuity etc.
- Archimedian/ solvability axioms
- Fineness/tightness

Invariance

- '$\succeq$ is preserved under some family of transforms.'

Shape

- 'Upper contour sets of $\succeq$ have certain shape'

What Are Invariance Axioms?

Definition

A binary relation $R \subseteq X \times X$, with asymmetric component P, is **invariant** under a transformation $\omega : X \to X$ if, for all $x, y \in X$:

$$x \, R \, y \quad \Longrightarrow \quad \omega(x) \, R \, \omega(y),$$

and

$$x \, P \, y \quad \Longrightarrow \quad \omega(x) \, P \, \omega(y).$$

Note: If R is invariant under ω, ω', then it is also invariant under $\omega \circ \omega'$ and $\omega' \circ \omega$.

- → Collection of transformations leaving R invariant always forms *semigroup* under $\circ$.
- → If R is invariant under every transformation in some semigroup of transformations $\mathcal{M}$, we say it is $\mathcal{M}$-invariant.

Examples I

Quasilinearity: $X = \mathbb{R}_+ \times Z$.

- For all $\alpha \geq 0$:

$$(t, z) \succsim (t', z') \quad \Longleftrightarrow \quad (t+\alpha, z) \succsim (t'+\alpha, z').$$

- See also:
  → Stationarity for dated rewards, translation invariance of utility functionals etc.

Homotheticity: $X =$ cone in vector space

- For all $\alpha > 0$:

$$x \succsim y \quad \Longleftrightarrow \quad \alpha x \succsim \alpha y.$$

- See also:
  → Cobb-Douglas: for all $(\alpha_1, \ldots, \alpha_K) \in \mathbb{R}^K_{++}$, and $x, y \in \mathbb{R}^K_+$,

$$(x_1, \ldots, x_K) \succsim (y_1, \ldots, y_K) \quad \Longleftrightarrow \quad (\alpha_1 x_1, \ldots, \alpha_K x_K) \succsim (\alpha_1 y_1, \ldots, \alpha_K y_K).$$

  → Constant Relative Risk Aversion: for all $\lambda > 0$, and $X, Y \in L^\infty$,

$$X \succsim Y \quad \Longleftrightarrow \quad \lambda X \succsim \lambda Y.$$

Examples II

Independence/Mixture Invariance: X is mixture space

- vNM Independence: for all $\alpha \in (0,1]$, and $\eta \in X$,

$$\mu \succsim \nu \quad \Longleftrightarrow \quad \alpha\mu + (1-\alpha)\eta \succsim \alpha\nu + (1-\alpha)\eta.$$

- See also:
  - → $*$-independence axioms for Anscombe-Aumann acts, dilutions of Blackwell experiments à la (Pomatto et al '23) etc.

Stationarity: $X = Z^{\mathbb{N}}$

- For all $z \in Z$:

$$(x_1, \ldots) \succsim (y_1, \ldots) \quad \Longleftrightarrow \quad (z, x_1, \ldots) \succsim (z, y_1, \ldots).$$

Examples III

Convolution Invariance: $X =$ lotteries on $\mathbb{R}$ with bounded support

- For all $\eta \in X$:

$$\mu \succsim \nu \quad \Longleftrightarrow \quad \mu * \eta \succsim \nu * \eta.$$

- See also:
  - $\rightarrow$ Constant Absolute Risk Aversion: for all $\alpha \in \mathbb{R}$

$$\mu \succsim \nu \quad \Longleftrightarrow \quad \mu * \delta_\alpha \succsim \nu * \delta_\alpha.$$

Products: $X =$ Blackwell experiments for finite set of states of the world Θ

- For all $(T, \{\eta_\theta\}_{\theta \in \Theta}) \in X$:

$$(S, \{\mu_\theta\}_{\theta\in\Theta}) \succsim (S', \{\nu_\theta\}_{\theta\in\Theta}) \quad \Longleftrightarrow \quad (S \times T, \{\mu_\theta \otimes \eta_\theta\}_{\theta\in\Theta}) \succsim (S' \times T, \{\nu_\theta \otimes \eta_\theta\}_{\theta\in\Theta}),$$

where $\succsim$ denotes 'more costly.'

Existing Work

- **Richter ('66 ECMA)**: Rationality (general environments)
- Afriat ('67 IER): Rational, monotone, convex, continuous preferences (linear budgets)
- Nishimura, Ok, Quah ('17 AER): Rational, monotone, continuous preferences (general topological environments)

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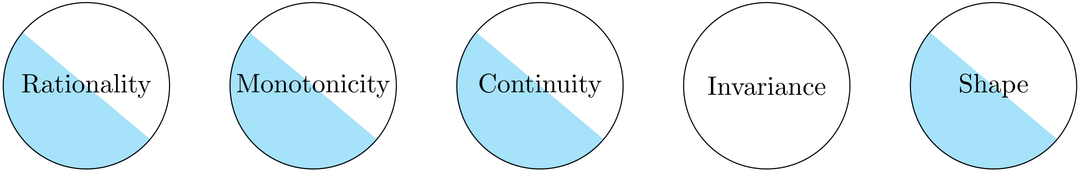

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Today: Rational preferences with arbitrary monotonicity/invariance axioms, on arbitrary environments.

Preliminaries

Let X be a set of alternatives, and $\mathcal{M}$ a given collection of transformations $X \to X$.

We assume as data a pair of observed **revealed preference** relations $\langle \succsim_R, \succ_R \rangle$.

- The relation $\succsim_R$ is '*revealed preferred*,' and $\succ_R$ is '*revealed strictly preferred*.'
- Focus on relations allows us to abstract from details of choice.
- Able to straightforwardly include arbitrary monotonicity requirements.

Primitives: X, $\mathcal{M}$, and $\langle \succsim_R, \succ_R \rangle$. We assume only $\mathrm{id} \in \mathcal{M}$, that $\mathcal{M}$ is $\circ$- closed and that $\succsim_R$ is reflexive.

Order Pairs

Definition

An **order pair** $\langle R, P \rangle$ is a pair of binary relations $R, P \subseteq X \times X$, such that $P \subseteq R$.

- Any binary relation $\succeq$ can be regarded as an order pair $\langle \succeq, \succ \rangle$, where we'll use $\succ$ to denote $\mathrm{Asymm}(\succeq)$.
- However, sometimes helpful to consider order pairs where P is not necessarily the asymmetric part of R, e.g. $\langle \succsim_R, \succ_R \rangle$.

Extending Binary Relations

Definition

An order pair $\langle R', P' \rangle$ **extends** $\langle R, P \rangle$ if: (i) $R \subseteq R'$, and (ii) $P \subseteq P'$.

Primary Question: When can the data $\langle \succsim_R, \succ_R \rangle$ be extended into an $\mathcal{M}$-invariant preference relation $\succeq$?

→ Existence of extending preference $\iff$ rationalizable (à la Richter).

→ Patterns which preclude existence of extension are *falsifiable predictions* of the model.

Notational Interlude

Notational Convention

We will use the following notation:

(i) **Compositions**: We denote $\omega \circ \omega'$ by juxtaposition, i.e. $\omega\omega'$.

(ii) **Transformations**: We denote $\omega(x)$ also by juxtaposition, i.e. ωx.

(iii) **Singleton sets**: When writing $\{(x, y)\}$, we omit curly braces, i.e. (x, y).

The Fundamental Difficulty: Knock-on Effects

Suppose $x, y \in X$ are $\succsim_R$-unrelated.

Observation

If we want to add a relation, e.g. $x \succeq y$, we generally pick up infinitely many **knock-on effects**, e.g. $\omega x \succeq \omega y$ for each $\omega \in \mathcal{M}$.

→ Even when adding $x \succeq y$ alone does not, *these can create cycles*.

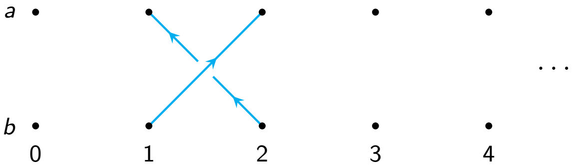
Knock-on Effects: An Example

Example

Let $X = \{a, b\} \times \{0, 1, 2, \ldots\}$, with $\mathcal{M} = \big\{(z, t) \mapsto (z, t + n)\big\}$, $n = 0, 1, \ldots$. Suppose we observe:

$$(a, 2) \succ_R (b, 1)$$
$$(a, 1) \succ_R (b, 2).$$

Under rationality, no restriction on the preference between $(a, 0), (b, 0)$. But every *stationary* rationalization must have $(a, 0) \succ (b, 0)$.



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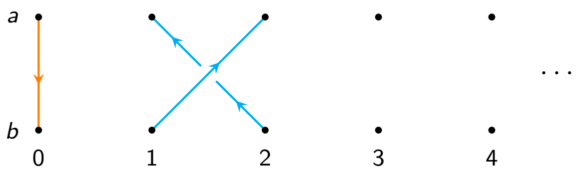

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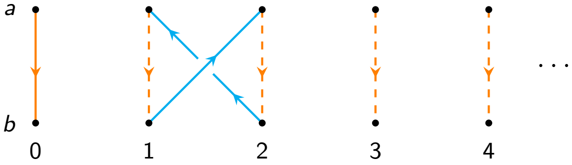

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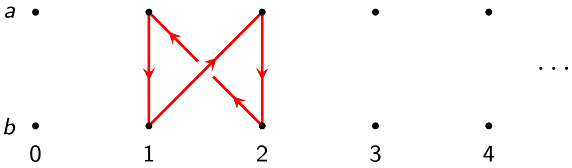


The $\mathcal{M}$-Closure

Definition

Given data $\langle \succsim_R, \succ_R \rangle$, define its $\mathcal{M}$-**closure** $\langle \succsim_R^{\mathcal{M}}, \succ_R^{\mathcal{M}} \rangle$ via:

$$\omega x \succsim_R^{\mathcal{M}} \omega y \quad \Longleftrightarrow \quad x \succsim_R y$$

and

$$\omega x \succ_R^{\mathcal{M}} \omega y \quad \Longleftrightarrow \quad x \succ_R y$$

Intuition: Just add all the 'translates' of pairs in $\langle \succsim_R, \succ_R \rangle$. Since $\mathrm{id} \in \mathcal{M}$, the $\mathcal{M}$-closure extends $\langle \succsim_R, \succ_R \rangle$.

Commutative Families: An Extension Theorem

Theorem

Let $\mathcal{M}$ be commutative, i.e. $\omega \circ \omega' = \omega' \circ \omega$ for all $\omega, \omega' \in \mathcal{M}$. Then the following are equivalent:

(i) The data $\langle \succsim_R, \succ_R \rangle$ are rationalizable by an $\mathcal{M}$-invariant preference relation.

(ii) The data's $\mathcal{M}$-closure $\langle \succsim_R^{\mathcal{M}}, \succ_R^{\mathcal{M}} \rangle$ is acyclic.

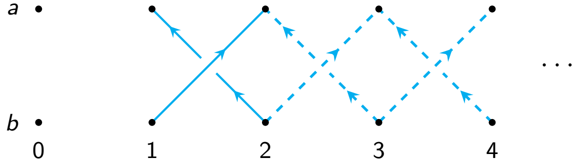
Proof Sketch

Big Picture: Classical transfinite induction argument...but trickier details.

Proof Sketch:

- If $\langle \succsim_R^{\mathcal{M}}, \succ_R^{\mathcal{M}} \rangle$ acyclic, commutativity allows us to straightforwardly extend data to invariant preorder.
- Show that if x, y are incomparable in this preorder, there exists an invariant preorder extension which ranks this pair.
  - → Invariance and commutativity imply that if no such extension exists, $\langle \succsim_R^{\mathcal{M}}, \succ_R^{\mathcal{M}} \rangle$ must contain a cycle...but this cycle may be *very* large/complicated.
- Standard Zorn's lemma argument provides maximal transitive, invariant extension, which must necessarily be complete by the preceding step. □

Stationary Extensions: Revisited

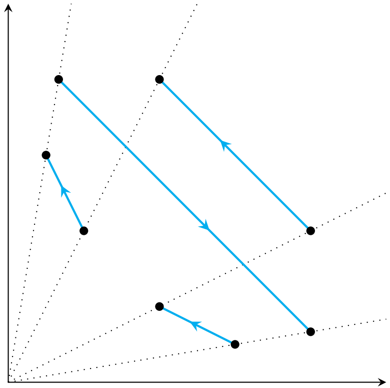


Observation: $\langle \succsim_R^{\mathcal{M}}, \succ_R^{\mathcal{M}} \rangle$ is acyclic — thus there exist stationary rationalizations!

Example: An $\mathcal{M}$-Closure Cycle

Example

Let $X = \mathbb{R}_+^N$, and $\mathcal{M}$ consist of the maps $x \mapsto \lambda x$, with $\lambda > 0$.



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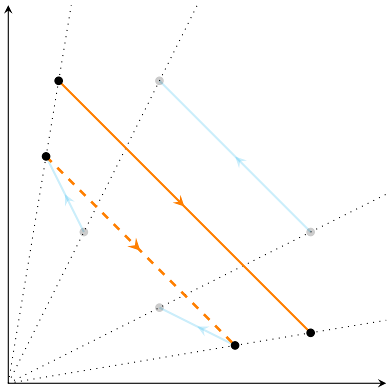

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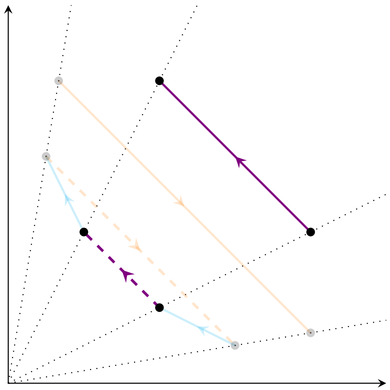

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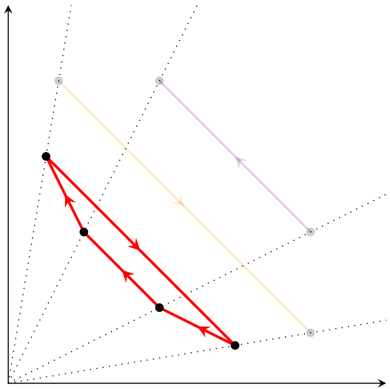


What's New Here

(i) Characterization of testable implications for some models where we had none, even via Afriat-type results.

→ General Fishburn-Rubinstein preferences, compactly supported monetary lotteries under convolution, general CARA/CRRA, etc. Dilution-invariant/Blackwell-monotone costliness orderings for experiments.

→ Even simple things like general (i.e. not necc. monotone) quasilinear or homothetic preferences.

(ii) Characterization of testable implications of classical models but for data from arbitrary budgets:

→ Monotone and quasilinear/homothetic/translation-invariant preferences etc.

Application: Probabilistic Sophistication

Definition

Let S be finite set of states of the world, and $X = 2^S$. A preference $\succsim$ on X is a **qualitative probability** if:

$$A \succsim B \iff A \cup C \succsim B \cup C,$$

for all events A, B and C disjoint from $A \cup B$.

We say a qualitative probability is *probabilistically sophisticated* if it can be represented by some measure in $\Delta(S)$.

Question: When can a qualitative probability be represented by a probability measure?

Orders on Functions

Let $X^* = \mathbb{Z}^S$ denote the set of all integer-valued functions on S, and let $\mathcal{M}$ denote the set of transformations on X^* of the form $f \mapsto f + g$, for $g \in X^*$.

- Any qualitative probability *induces* a transitive (but incompete) order $\succsim^*$ on X^* via:

$$A \succsim B \quad \Longleftrightarrow \quad \mathbb{1}_A \succsim^* \mathbb{1}_B.$$

- Any probability measure $\mu \in \Delta(S)$ induces an (i) complete, (ii) transitive, (iii) monotone, and (iv) $\mathcal{M}$-invariant ordering $\succeq$ on X^* via:

$$f \succeq g \quad \Longleftrightarrow \quad \int f \, d\mu \geq \int g \, d\mu.$$

- However, not every order satisfying (i) - (iv) has such a representation...

A Simple (New) Characterization

The following is a straightforward consequence of Theorem 1.4 in Scott (1964).

Proposition

A qualitative probability $\succsim$ on X is probabilistically sophisticated if and only if $\succsim^*$ can be extended to an $\mathcal{M}$-invariant preference on X^*.

Thus:

Corollary

A qualitative probability $\succsim$ is representable by a probability measure if and only if the $\mathcal{M}$-closure of $\succsim^*$ is acyclic.

Connections to Afriat-Type Theories

Assumption

Suppose that $\langle \succsim_R, \succ_R \rangle$ is obtained from price-consumption data.

In the Paper: Show our acyclicity condition on $\langle \succsim_R^{\mathcal{M}}, \succ_R^{\mathcal{M}} \rangle$ reduces to standard, model-specific GARP variations from literature.

→ E.g. HARP (Varian '83), cyclic monotonicity (Brown & Calsamiglia '07) etc.

Without Commutativity, All Bets Are Off

Example

Let Z be a space of prizes, and $X = Z^{\mathbb{N}}$. Let $\mathcal{M}$ consist of all transformations of the form:

$$(x_1, x_2, \ldots) \quad \mapsto \quad (z, x_1, \ldots)$$

for some $z \in Z$. Suppose we observe $\succ_R$ given by:

$$(a, x_1, \ldots) \succ_R (b, y_1, \ldots) \qquad (c, y_1, \ldots) \succ_R (d, x_1, \ldots)$$
$$(b, x_1, \ldots) \succ_R (a, y_1, \ldots) \qquad (d, y_1, \ldots) \succ_R (c, x_1, \ldots)$$

for $a, b, c, d \in Z$, and $x, y \in X$. Note (i) that $\succ_R$ is transitive, and (ii) $\succ_R^{\mathcal{M}}$ is acyclic.

What Went Wrong?

Intuition: Something similar to the Fishburn-Rubinstein example goes wrong.

→ Adding $y \succeq x$ yields knock-on effects (i) $ay \succeq ax$, and (ii) $by \succeq bx$. But then:

$$ax \succ_R by \succeq bx \succ_R ay \succeq ax.$$

→ But, analogously, adding $x \succeq y$ also creates a cycle:

$$cy \succ_R dx \succeq dy \succ_R cx \succeq cy.$$

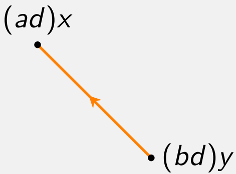
Non-Commutativity To Blame

Example

Suppose we allow ourselves to pass the transforms a, b, c, d through each other. Recall:

$$ax \succ_R by \qquad bx \succ_R ay \qquad cy \succ_R dx \qquad dy \succ_R cx.$$

Then in the $\mathcal{M}$-closure:



$$\begin{aligned} ax &\succ_R^{\mathcal{M}} by && \text{HYP} \\ \Longrightarrow (da)x &\succ_R^{\mathcal{M}} (db)y && \text{INV} \\ \Longrightarrow (ad)x &\succ_R^{\mathcal{M}} (bd)y && \text{COM} \end{aligned}$$

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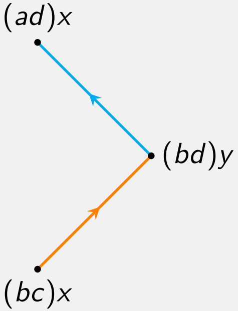

$$\begin{aligned} & dy \succ_R^{\mathcal{M}} cx && \text{HYP} \\ \Longrightarrow\ & (bd)y \succ_R^{\mathcal{M}} (bc)x && \text{INV} \end{aligned}$$

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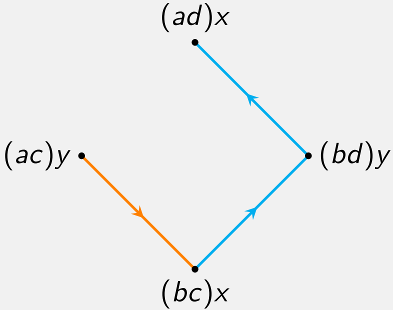

$$\begin{aligned} & bx \succ_R^{\mathcal{M}} ay && \text{HYP} \\ \Longrightarrow\ & (cb)x \succ_R^{\mathcal{M}} (ca)y && \text{INV} \\ \Longrightarrow\ & (bc)x \succ_R^{\mathcal{M}} (ac)y && \text{COM} \end{aligned}$$

Non-Commutativity To Blame

Example

Suppose we allow ourselves to pass the transforms a, b, c, d through each other. Recall:

$$\cancel{ax \succ_R by} \qquad \cancel{bx \succ_R ay} \qquad cy \succ_R dx \qquad \cancel{dy \succ_R cx}.$$

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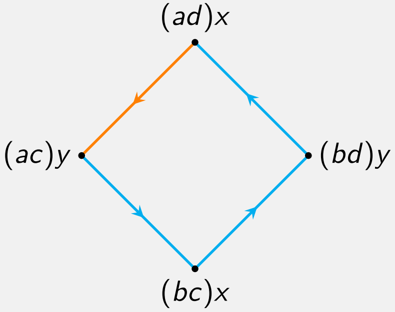


$$\begin{aligned} & cy \succ_R^{\mathcal{M}} dx && \text{HYP} \\ \Longrightarrow\ & (ac)y \succ_R^{\mathcal{M}} (ad)x && \text{INV} \end{aligned}$$

Broken Cycles

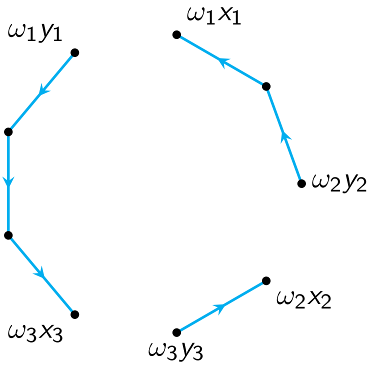
Definition

We say $\omega_1, \ldots, \omega_N \in \mathcal{M}$ and $x_1, y_1, \ldots, x_N, y_N \in X$ define a **broken cycle** if:

$$\begin{aligned}
\omega_1 x_1 &\succsim_R^{\mathsf{T}} \omega_2 y_2 \\
\omega_2 x_2 &\succsim_R^{\mathsf{T}} \omega_3 y_3 \\
&\vdots \qquad \vdots \\
\omega_N x_N &\succsim_R^{\mathsf{T}} \omega_1 y_1,
\end{aligned}$$

and x_i is not comparable to y_i, for all $1 \leq i \leq N$. If any $\succsim_R^{\mathsf{T}}$ sequence contains a $\succ_R$, we call it a *strict* broken cycle.

Intuition



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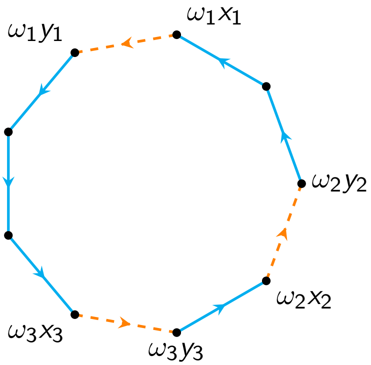


...and Forbidden Subrelations

Suppose we have a broken cycle:

$$\begin{aligned} \omega_1 x_1 &\succsim_R^\top \omega_2 y_2 \\ \omega_2 x_2 &\succsim_R^\top \omega_3 y_3 \\ &\vdots \\ \omega_N x_N &\succsim_R^\top \omega_1 y_1, \end{aligned} \tag{$*$}$$

Definition

An order pair $\langle F, G \rangle$ is a **forbidden subrelation** obtained from $(*)$ if:

(i) The relation $F = \{(y_1, x_1), \ldots, (y_N, x_N)\}$; and

(ii) If $(*)$ is not strict, then $\varnothing \subsetneq G \; (\subseteq F)$.

Intuition

Subrelations as Restrictions: Suppose $\langle F, G \rangle$ is a forbidden subrelation. If a binary relation $\succeq$ extends it then:

(i) Every pair in F belongs to $\succeq$; and

(ii) Every pair in G belongs to $\succ$.

But this means $\succeq$ completes the broken cycle which generated $\langle F, G \rangle$ $\rightarrow$ can't be a preference.

$\rightarrow$ Forbidden subrelations capture *set-valued* restrictions on the extension problem.

A Necessary Condition:

When can we extend the data $\langle \succsim_R, \succ_R \rangle$ while *not* extending any forbidden subrelations?

Indirect Restrictions

Example

Suppose we have two forbidden subrelations $\langle F_1, \varnothing \rangle$ and $\langle F_2, \varnothing \rangle$, where:

$$F_1 = \{(x, y), (y', x')\} \quad \text{and} \quad F_2 = \{(y, x), (y'', x'')\}.$$

Any rationalizing preference $\succeq$ can't extend either F_1 or F_2. But it must rank $x \succeq y$ or $y \succeq x$ — which means it also can't extend:

$$\tilde{F} = \big(F_1 \setminus (x, y)\big) \cup \big(F_2 \setminus (y, x)\big).$$

The relation $\tilde{F}$ encodes an *indirect* restriction to the extension problem.

The ‘Collapse’

Definition

Given finite order pairs $\langle F_1, G_1 \rangle$, and $\langle F_2, G_2 \rangle$, we say an order pair $\langle \tilde{F}, \tilde{G} \rangle$ is their **collapse** if:

(i) For some $\omega, \omega' \in \mathcal{M}$ and $x, y \in X$,

$$\big(\omega x, \omega y\big) \in F_i \setminus G_i \quad \text{and} \quad \big(\omega' y, \omega' x\big) \in F_j,$$

where $i \neq j$.

(ii) The relations $\tilde{F}$ and $\tilde{G}$ are given by:

$$\tilde{F} = \big(F_i \setminus (\omega x, \omega y)\big) \, \cup \, \big(F_j \setminus (\omega' y, \omega' x)\big)$$

and

$$\tilde{G} = G_i \, \cup \, \big(G_j \setminus (\omega' y, \omega' x)\big).$$

Generating Restrictions: New and Old

Collapse:

$\{(\omega x, \omega y), (x', y'), \ldots\}$ $\{(\omega' y, \omega' x), (x'', y''), \ldots\}$

$\{(x', y'), (x'', y''), \ldots\}$

Cancel out 'clashing pair.'

Transitive Closure:

(x, y) (y, z)

(x, z)

Cancel out 'clashing alternative.'

Strong Acyclicity

Let $\mathcal{F}^0$ denote the set of all forbidden subrelations generated by some broken cycle in the data.

Define: For all $n \geq 1$,

$$\mathcal{F}^n = \left\{\langle F, G\rangle : \langle F, G\rangle \text{ is collapse of pairs in } \mathcal{F}^{n-1}\right\} \cup \mathcal{F}^{n-1}.$$

Let:

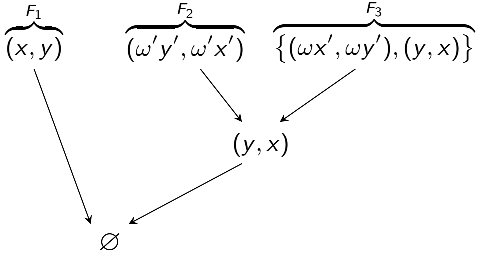
$$\mathcal{F}^* = \bigcup_{n\geq 1} \mathcal{F}^n.$$

Definition

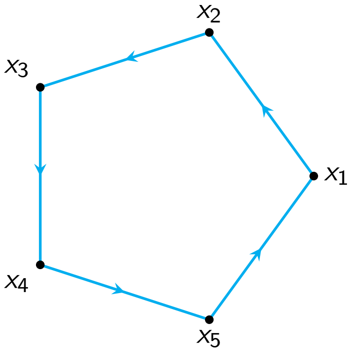
We say that $\langle \succsim_R, \succ_R\rangle$ is **strongly acyclic** if $\langle \varnothing, \varnothing\rangle \notin \mathcal{F}^*$.

Cycles: New and Old

Collapse: A ‘cycle’ is a collection of order pairs where every *relation* cancels, e.g. $G_1 = G_2 = G_3 = \varnothing$, and:



Transitive closure: A cycle is a set of pairs where every *alternative* cancels.



Throwback: A Violation of Strong Acyclicity

Example

Suppose again that we've observed:

$$(a, x_1, \ldots) \succ_R (b, y_1, \ldots) \qquad (c, y_1, \ldots) \succ_R (d, x_1, \ldots)$$
$$(b, x_1, \ldots) \succ_R (a, y_1, \ldots) \qquad (d, y_1, \ldots) \succ_R (c, x_1, \ldots)$$

for $a, b, c, d \in Z$, and $x, y \in X$. These are broken cycles, with forbidden subrelations:

$$\big\langle (y, x), \varnothing \big\rangle \qquad \text{and} \qquad \big\langle (x, y), \varnothing \big\rangle.$$

Their collapse is $\langle \varnothing, \varnothing \rangle$, hence $\langle \succsim_R, \succ_R \rangle$ is not strongly acyclic!

Characterizing Invariant Rationalizability

Theorem

The following are equivalent:

(i) The data $\langle \succsim_R, \succ_R \rangle$ are rationalizable by an $\mathcal{M}$-invariant preference relation.

(ii) The data are strongly acyclic.

Note: Requires no assumptions on X, $\mathcal{M}$, or $\langle \succsim_R, \succ_R \rangle$.

Proof Sketch: Preliminaries

Idea: Re-encode problem in terms of *propositional logic*.

- For all $(x, y) \in X \times X$, we define two boolean variables:

$$[\mathrm{x} \succeq \mathrm{y}] \quad \text{and} \quad [\mathrm{x} \succ \mathrm{y}].$$

- We denote the collection of all these variables by $\mathcal{V}$.
- Introduce *formulas* relating these variables, so that there is a 1-1 correspondence between assignments of $\{\top, \bot\}$ satisfying these formulas, and invariant rationalizations of $\langle \succsim_R, \succ_R \rangle$.

Proof Sketch: Preliminaries

(i) **Completeness**: For each $x, y \in X$:

$$[\mathrm{x} \succeq \mathrm{y}] \vee [\mathrm{y} \succeq \mathrm{x}].$$

(ii) **Coherency**: For each $x, y \in X$, we have two formulas:

$$\neg[\mathrm{x} \succeq \mathrm{y}] \vee \neg[\mathrm{y} \succ \mathrm{x}],$$

and

$$[\mathrm{x} \succeq \mathrm{y}] \vee [\mathrm{y} \succ \mathrm{x}].$$

(iii) **Transitivity**: For all $x, y, z \in X$:

$$\neg[\mathrm{x} \succeq \mathrm{y}] \vee \neg[\mathrm{y} \succeq \mathrm{z}] \vee [\mathrm{x} \succeq \mathrm{z}].$$

(iv) **Extension**: For all $(x, y) \in \succsim_R$:

$$[\mathrm{x} \succeq \mathrm{y}],$$

and for all $(x, y) \in \succ_R$:

$$[\mathrm{x} \succ \mathrm{y}].$$

(v) **Invariance**: For all $x, y \in X$ and $\omega \in \mathcal{M}$:

$$\neg[\mathrm{x} \succeq \mathrm{y}] \vee [\omega\mathrm{x} \succeq \omega\mathrm{y}],$$

and

$$[\mathrm{x} \succeq \mathrm{y}] \vee \neg[\omega\mathrm{x} \succeq \omega\mathrm{y}].$$

Conversion Lemma

Let Φ denote the collection of all formulas of form (i) - (v).

Lemma

There exists an $\mathcal{M}$-invariant preference rationalizing $\langle \succsim_R, \succ_R \rangle$ if and only if Φ is satisfiable.

Interlude: Propositional Resolution

Suppose A_1, A_2, A_3 are *literals*, i.e. each equal to V_i or $\neg V_i$ for some $V_i \in \mathcal{V}$, and consider the clauses:

$$C = A_1 \vee A_2 \quad \text{and} \quad C' = \neg A_1 \vee A_3.$$

Observation

If C and C' evaluate to true for some assignment of truth values to the underlying variables, so must:

$$D = A_2 \vee A_3,$$

as either A_1 or $\neg A_1$ must be true.

Interlude: Propositional Resolution

More generally, let $A_1, \ldots, A_K$, $B_1 \ldots, B_L$ be literals, where $A_1 = \neg B_1$, and consider the clauses:

$$C = \bigvee_{k=1}^{K} A_k \quad \text{and} \quad C' = \bigvee_{l=1}^{L} B_l.$$

If C and C' evaluate to true for some assignment of truth values, then so must:

$$D = \left[\bigvee_{k=2}^{K} A_k\right] \vee \left[\bigvee_{l=2}^{L} B_l\right].$$

Definition

The clause D is called the **resolvent** of C, C', and $C \wedge C'$ is logically equivalent to $C \wedge C' \wedge D$.

A Consequence

Suppose we have two clauses:

$$C = A_1 \quad \text{and} \quad C' = \neg A_1.$$

Their resolvent is the empty clause, $\varnothing$, which is always *false*. Then $C \wedge C'$ is logically equivalent to $C \wedge C' \wedge \varnothing$, which is unsatisfiable, hence so is $C \wedge C'$.

Takeaway: If, through finitely many resolution steps, we can 'derive' the empty clause, the original collection of clauses must be unsatisfiable.

Proof Sketch: Necessity

Lemma

Suppose $\langle \succsim_R, \succ_R \rangle$ is not strongly acyclic. Then Φ is unsatisfiable.

Proof Sketch:

- Every $\langle F, G \rangle \in \mathcal{F}^0$ can be expressed uniquely as disjunction of negative literals:

$$C_{FG} = \left[\bigvee_{(x,y) \in F \setminus G} \neg[\mathrm{x} \succeq \mathrm{y}] \right] \vee \left[\bigvee_{(x,y) \in G} \neg[\mathrm{x} \succ \mathrm{y}] \right].$$

  Every such C_{FG} can be obtained from Φ purely via resolution.

- If $\langle \bar{F}, \bar{G} \rangle$ is the collapse of $\langle F_1, G_1 \rangle$ and $\langle F_2, G_2 \rangle$, $C_{\bar{F}\bar{G}}$ can be obtained iteratively through resolution steps involving $C_{F_1 G_1}$, $C_{F_2 G_2}$ and clauses in Φ.

- Thus the empty clause can be obtained via resolution from Φ, which is unsatisfiable. Thus Φ is unsatisfiable. □

Robinson's Theorem

Theorem (Robinson, 1965)

A finite set of clauses Φ' is unsatisfiable if and only if the empty clause $\varnothing$ can be obtained from Φ' through repeated resolution steps.

We rely on a slight strengthening of this result, which is standard in computer science.

Theorem (Negative Resolution Theorem)

A finite set of clauses Φ' is unsatisfiable if and only if the empty clause $\varnothing$ can be obtained from Φ' through repeated resolution, where every step involves a parent clause with no positive literals.

Takeaway: 'Negative resolution,' as a proof strategy, is *refutation-complete*.

Proof Sketch: Sufficiency

Lemma

Suppose Φ is unsatisfiable. Then $\langle \succsim_R, \succ_R \rangle$ is not strongly acyclic.

Proof Sketch:

- By Propositional Compactness (i.e. Tychonoff's theorem), if Φ is unsatisfiable there is a finite, unsatisfiable subset Φ'.
- By the Negative Resolution Theorem, there exists a binary proof tree which derives the empty clause from clauses in Φ' purely via negative resolution.
- Each node on such a proof tree corresponding to a clause with no positive literals is the clausal representation of an order pair in some $\mathcal{F}^n$, where n depends on the node's depth in the tree.
- In particular, the empty order pair $\langle \varnothing, \varnothing \rangle$ belongs to some $\mathcal{F}^n$ and hence $\mathcal{F}^*$. □

Application: Expected Utility

Let X denote the set of probability distributions over some finite prize space, and $\mathcal{M}$ denote all transformations of the form:

$$\mu \mapsto \alpha\mu + (1-\alpha)\nu$$

for some $\alpha \in (0,1]$ and $\nu \in X$.

Theorem

Suppose $\succsim_R$ is *finite*. Then the following are equivalent:

(i) $\langle \succsim_R, \succ_R \rangle$ is strongly acyclic.

(ii) The data are rationalized by an expected utility preference.

Takeaway: When $\langle \succsim_R, \succ_R \rangle$ finite, able to obtain *continuous* invariant rationalizations.

Extension: Out-of-Sample Predictions

Suppose the data $\langle \succsim_R, \succ_R \rangle$ are strongly acyclic ($\iff$ rationalizable) but x and y are $\succsim_R$-unrelated.

<u>**Question**</u>: When does *every* invariant rationalization agree on their ranking?

The Dushnik-Miller Theorem

Classically, i.e. in the case when $\mathcal{M} = \{\mathrm{id}\}$, the answer is not particularly interesting...

Theorem (Dushnik & Miller)

Suppose $\succsim_R$ is an acyclic binary relation with strict component $\succ_R$. Then:

$$\succsim_R^{\mathsf{T}} = \bigcap_{\succeq \, \in \mathcal{P}(\succsim_R)} \succeq,$$

where $\succsim_R^{\mathsf{T}}$ denotes the transitive closure of $\succsim_R$, and $\mathcal{P}(\succsim_R)$ denotes the (non-empty) set of all preference relations extending $\succsim_R$.

Takeaway: Only out-of-sample predictions are given by $\succsim_R^{\mathsf{T}} \setminus \succsim_R$.

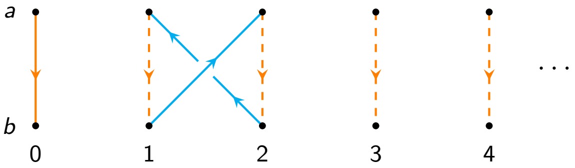
The Invariant Case: More Interesting?

Example

Let $X = \{a, b\} \times \{0, 1, 2, \ldots\}$. Suppose we observe:

$$(a, 2) \succ_R (b, 1)$$
$$(a, 1) \succ_R (b, 2).$$

Under rationality, no restriction on the preference between $(a, 0), (b, 0)$. But every *stationary* rationalization must have $(a, 0) \succ (b, 0)$.



Out of Sample Predictions: An Extreme Example

Suppose $X = \mathbb{R}^2_+$, and $\mathcal{M}$ consists of all transformations $(x_1, x_2) \mapsto (\lambda_1 x_1, \lambda_2 x_2)$, for $\lambda \gg 0$.

- Cobb-Douglas preferences are the unique (i) monotone, (ii) continuous, and (iii) $\mathcal{M}$-invariant preferences.

Example

Suppose for some $x, x' \gg 0$, where x, x' are $\geq$-incomparable, we observe $x \sim_R x'$. There is a *unique* Cobb-Douglas preference consistent with $\succsim_R$.

Takeaway: By considering more structured rationalizations, obtain (possibly *much*) richer out-of-sample predictions.

Characterizing of Out-of-Sample Predictions

Theorem

Suppose $\langle \succsim_R, \succ_R \rangle$ is strongly acyclic. Then $x \succeq^* y$ (resp. $x \succ^* y$) for every $\mathcal{M}$-invariant rationalization $\succeq^*$ if and only if:

$$\big\langle (y,x),(y,x) \big\rangle \in \mathcal{F}^* \quad \big(\text{resp.} \big\langle (y,x), \varnothing \big\rangle \in \mathcal{F}^* \big).$$

<u>**Takeaway**</u>: Every rationalizing preference ranks x over y if and only if the opposite relation arises as a 'singleton' restriction.

Proof Sketch

Proof Sketch:

- Clearly if $\mathcal{F}^*$ contains some singleton restriction, then any restriction must satisfy it and hence agree on that pair.
- Conversely, suppose every extension agrees $x \succeq^* y$. Then every valid model for Φ evaluates $[\mathtt{y} \succ \mathtt{x}]$ to $\perp$.
- Define $\tilde{\Phi}$ from Φ by first:
  (i) Removing any clause containing $[\mathtt{y} \succ \mathtt{x}]$; and
  (ii) Deleting $\neg[\mathtt{y} \succ \mathtt{x}]$ from any remaining clause which contains it.
- By construction, there is a 1-1 correspondence between models for Φ that assigns $[\mathtt{y} \succ \mathtt{x}]$ to $\top$ and models for $\tilde{\Phi}$, hence $\tilde{\Phi}$ is unsatisfiable.

Proof Sketch: Continued

Proof Sketch (Cont'd):

- By compactness, there exists a finite unsatisfiable subset $\tilde{\Phi}' \subset \tilde{\Phi}$, and a derivation of $\varnothing$ from $\tilde{\Phi}'$ via negative resolution.
- Every clause D in the proof tree that contains no positive literals either (i) can be derived from Φ via NR, or (ii) $D \vee \neg[\mathrm{y} \succ \mathrm{x}]$ can be derived from Φ via NR.
- Since Φ is satisfiable by hypothesis, $\varnothing$ cannot be obtained from Φ in this way, thus $\varnothing \vee \neg[\mathrm{y} \succ \mathrm{x}] = \neg[\mathrm{y} \succ \mathrm{x}]$ can be.
- As $\neg[\mathrm{y} \succ \mathrm{x}]$ contains no positive literals and is deduced from Φ via negative resolution, $\big\langle (y, x), (y, x) \big\rangle$ can be obtained via collapses from $\mathcal{F}^0$. An analogous argument holds for $x \succ^* y$. $\square$

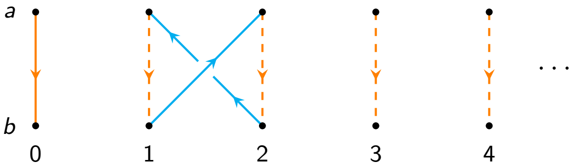
Example Revisited

Example

Let $X = \{a, b\} \times \{0, 1, 2, \ldots\}$. Suppose we observe:

$$(a, 2) \succ_R (b, 1)$$
$$(a, 1) \succ_R (b, 2).$$

Under rationality, no restriction on the preference between $(a, 0), (b, 0)$. But every *stationary* rationalization must have $(a, 0) \succ (b, 0)$.



Extension: Invariance Under Partial Functions

- **Additive Separability/P2**: $X = \mathcal{X}^S$, where $|S| > 2$. For all $A \subseteq S$ and $x, y, z, z' \in \mathcal{X}$:

$$(x_A z) \succeq (y_A z) \iff (x_A z') \succeq (y_A z').$$

  → For each $B \subseteq S$ and acts $\hat{z}, \hat{z}'$ on B, have map that takes all acts equal to $\hat{z}$ on B and replaces them with $\hat{z}'$.

- **Qualitative Probabilities**: $X = \mathcal{A}$, an algebra of subsets of S.

$$A \succeq B \iff A \cup C \succeq B \cup C$$

  for any $C \in \mathcal{A}$ disjoint from A, B.

  → For each $C \in \mathcal{A}$ map that takes union with C, but whose domain is precisely those sets disjoint from C.

- **Many More**: Comonotonic additivity for CEU, sign-comonotonic consistency for CPT (Wakker & Tversky '93) etc.

The Longer Term

Long Term Objective

A 'modular,' 'universal' revealed preference theory: "if your axioms fall into bins A, B and C, then the testable implications are ____."

Long-run Objective: Write the 'last revealed preference theorem.'

Conclusion

- Many of preference properties of first-order economic importance are *invariance* axioms.
- Historically, no unified theory of testable implications. Reliance on ad hoc methods in special cases.
- **This Paper**: Characterization of the testable implications of *any* axiom of this form, for any revealed preference data, on any domain. Characterization of the out-of-sample properties generated by any such axioms.
  - → Novel methodological approach that lends itself to further generalizations applying tools from computer science and logic.

Thank You!

Any Questions?